

Review

# Automorphisms and autoequivalences of generic analytic $K3$ surfaces

Emanuele Macrì<sup>a,\*</sup>, Paolo Stellari<sup>b</sup>

<sup>a</sup>*Hausdorff Center for Mathematics, Mathematisches Institut, Universität Bonn, Beringstrasse 1, 53115 Bonn, Germany*

<sup>b</sup>*Dipartimento di Matematica “F. Enriques”, Università degli Studi di Milano, Via Cesare Saldini 50, 20133 Milano, Italy*

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## Abstract

This is a systematic exposition of recent results which completely describe the group of automorphisms and the group of autoequivalences of generic analytic  $K3$  surfaces. These groups, hard to determine in the algebraic case, admit a good description for generic analytic  $K3$  surfaces, and are in fact seen to be closely interrelated.

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\* Corresponding author. Tel.: +49 (0)2287362204.

*E-mail addresses:* [macri@math.uni-bonn.de](mailto:macri@math.uni-bonn.de) (E. Macrì), [Paolo.Stellari@mat.unimi.it](mailto:Paolo.Stellari@mat.unimi.it) (P. Stellari).

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## 1. Introduction

In this survey paper we present some recent results about the group of automorphisms and the group of autoequivalences of generic analytic  $K3$  surfaces. Recall that a surface of this type is a non-algebraic  $K3$  surface with trivial Picard group. They are dense in the moduli space of  $K3$  surfaces and their geometry is considerably simple when compared to that of algebraic  $K3$  surfaces. This is mainly due to the lack of curves.

The geometry of  $K3$  surfaces has been widely studied during the last forty years with particular attention to the algebraic case. Deep results such as the Torelli Theorem have shown that many interesting geometric properties of these surfaces can be determined by lattice theory and Hodge structures. For example, the cup product and the weight-two Hodge structure defined on the second cohomology group of a  $K3$  surface  $X$  determine it uniquely up to isomorphisms. Thus it forms a key tool in the study of the automorphism groups of  $K3$  surfaces.

The algebraic case happens to be extremely rich and, in fact, we do not have a complete description of the automorphism group of algebraic  $K3$  surfaces at this point. It turns out that the isomorphism class of the Picard lattice can deeply influence this group. One can easily produce examples where this group is trivial, finite, infinite and cyclic or infinite but not cyclic (Section 3.2). Very often the geometry of the surface and not just the lattice and Hodge structures of its second cohomology group gives significant clues to approach a description of this group. From this intricate situation, a quite large number of interesting results emerged in the algebraic setting.

The case of generic analytic  $K3$  surfaces is very different. In particular the description of the automorphism group is very neat and only two possibilities occur: either the group is trivial or the group is cyclic and infinite (see Theorem 3.6 and [70]). The latter case is quite rare and the generic analytic  $K3$  surfaces where this is realized have been completely described by Gross and McMullen (see Section 2.4 and [37,58]).

There has been a heightened interest in  $K3$  surfaces recently because of advances in string theory. Indeed, these surfaces are examples of Calabi–Yau manifolds and irreducible holomorphic symplectic manifolds at the same time and their geometry is relatively well-understood. Hence they have become one of the favorite examples to test the homological interpretation of Mirror Symmetry due to Kontsevich ([54] but see also [24,41,78]).

Roughly speaking, this conjecture predicts that, for two Calabi–Yau manifolds  $X$  and  $Y$  which are mirrors, the bounded derived category of coherent sheaves on one variety should be equivalent to the derived Fukaya category of the other one. The bounded derived category of coherent sheaves should take into account the complex structure of a variety while the derived Fukaya category should encode the symplectic geometry. Indeed the objects of the latter category are special Lagrangian submanifolds with a suitable notion of morphisms obtained by means of Floer homology.

It is known that the abelian category of coherent sheaves  $\mathbf{Coh}(X)$  on a smooth projective variety  $X$  determines the variety uniquely up to isomorphism. This is no longer true if we consider the bounded derived category of  $\mathbf{Coh}(X)$ . More precisely, Bondal and Orlov proved in [10] that if either the canonical or the anticanonical bundle of  $X$  is ample then  $X$  is determined uniquely up to isomorphism by the bounded derived category of  $\mathbf{Coh}(X)$ . But as soon as the canonical bundle becomes trivial (as for  $K3$  surfaces and, more generally, for Calabi–Yau manifolds), a large number of interesting phenomena can happen. Just restricting to the case of algebraic  $K3$  surfaces, one can produce examples of  $K3$  surfaces  $X$  and arbitrarily many non-isomorphic  $K3$  surfaces  $Y$  with the same derived category [69, 83]. Nevertheless the number of isomorphism classes of  $K3$  surfaces with such a close relationship to  $X$  is always finite [17]. Conjecturally the number of isomorphism classes of smooth algebraic varieties with the same derived categories is always expected to be finite.

One can wonder if the same results are still true when  $X$  is non-algebraic but we will see that this is not the case. As a first step, we will observe that, in order to be sure that all functors and dualities we need are well-defined, we should work with a suitable subcategory of the derived category of  $\mathcal{O}_X$ -modules instead of the bounded derived category

of coherent sheaves (Section 4.3). The main problem in this sense is the validity of Serre duality. Notice that in our geometric setting, just involving complex non-projective surfaces, this is not a stumbling block since, exactly as in the algebraic case, this special subcategory is naturally equivalent to the bounded derived category of coherent sheaves (see Proposition 4.15 and [9]) although some problems arise when we deal with the derived category of the product of two  $K3$  surfaces. Moreover, the geometric similarities with the algebraic setting are evident in many other useful instances. For example, any coherent sheaf admits a locally free resolution of finite length. Hence the Grothendieck group of these surfaces is generated by the classes of locally free sheaves, we can easily define Chern classes for coherent sheaves and the Riemann–Roch Theorem in its non-algebraic version holds true for any coherent sheaf. More generally we will recall how Grothendieck–Riemann–Roch Theorem was proved in [65] after Chern classes were introduced in [66] for coherent sheaves on analytic varieties. A somewhat sketchy definition of these Chern classes will be included in Section 4.1.

Assuming the point of view of physics one can see that many interesting dualities in string theory can be reinterpreted as equivalences (or autoequivalences) of derived categories. This motivates the growing interest in the study of the group of autoequivalences of the derived categories of  $K3$  surfaces or, more generally, of varieties with trivial canonical bundle.

While a very nice and complete description of the group of autoequivalences is now available for abelian varieties [74] the same is not true for algebraic  $K3$  surfaces. Even for very explicit examples where the geometry of the surface is very well-understood, this group remains mysterious.

A conjectural description of it was first proposed by Szendrői in [84]. Indeed, Orlov [73] proved, using results of Mukai [60], that any autoequivalence of the derived category of a smooth projective  $K3$  surface induces an isomorphism of the total cohomology groups preserving some special lattice and Hodge structures (Section 5.1). Szendrői observed that such an isomorphism should preserve the orientation of a particular four-dimensional subspace of the real cohomology (Conjecture 5.6). So we would expect a surjective morphism from the group of autoequivalences onto a group of orientation preserving isometries of the total cohomology group of an algebraic  $K3$  surface. Notice that this is in perfect analogy with the Strong Torelli Theorem where the preservation of the Kähler cone is required.

On the other hand, until recently no clue about the kernel of this morphism was available. The real breakthrough in this direction is due to Bridgeland and to his notion of stability condition on triangulated categories [12]. Roughly speaking a stability condition is determined by a  $t$ -structure on the triangulated category and a  $\mathbb{C}$ -linear function on the heart of this  $t$ -structure satisfying special properties which imitate the usual definition of  $\mu$ -stability for coherent sheaves. Bridgeland’s motivation was to axiomatize the notion of  $II$ -stability which emerged from Douglas’ work [27]. In particular, the manifold parametrizing stability conditions on the derived category  $\mathbf{D}$  of some Calabi–Yau manifold quotiented by the natural action of the autoequivalence group of  $\mathbf{D}$  models an extended version of the stringy Kähler moduli space of the Calabi–Yau manifold.

Studying stability conditions on the bounded derived category of coherent sheaves on smooth projective  $K3$  surfaces, Bridgeland conjectured that the kernel of the morphism considered by Szendrői should be the fundamental group of some period domain associated with the variety parametrizing stability conditions on the derived category [13].

An attempt to verify this conjectural description of the autoequivalence group due to Bridgeland and Szendrői was made in [45] where this group was described for generic twisted  $K3$  surfaces (not considered in this paper) and generic analytic  $K3$  surfaces. Very surprisingly, the groups of automorphisms and autoequivalences which are quite mysterious in the algebraic case, can be explicitly studied in the generic analytic case. Moreover it turns out that these groups are very closely related in the non-algebraic setting since the group of autoequivalences of Fourier–Mukai type of a generic analytic  $K3$  surface just consists of shifts, twists by a special object called spherical and automorphisms (Theorem 5.17). (Notice that the techniques used in [45] to prove this result, and briefly recalled in the second part of the present paper, are applied in [46] to prove Szendrői’s conjecture in the smooth projective case.)

The aim of this paper is to describe these two groups in full detail, collecting results contained in many papers and trying to give a comprehensive, systematic and integrated approach.

The paper is organized as follows. Section 2 deals with the geometry of  $K3$  surfaces. We briefly recall some results in lattice theory and give some examples of algebraic and non-algebraic  $K3$  surfaces. Particular attention is paid to the examples of non-algebraic  $K3$  surfaces proposed by McMullen. We also discuss the lattice and Hodge structures defined on the second and total cohomology groups of  $K3$  surfaces.

Section 3 is devoted to the complete description of the automorphism group of generic analytic  $K3$  surfaces. In particular we present the works of Gross, McMullen and Oguiso. These results are compared through examples with the case of algebraic  $K3$  surfaces.

In Section 4 we identify the appropriate triangulated category where all functors with geometrical meaning can be derived (as a special case we consider the Serre functor). We recall a result of Verbitsky about the abelian category of coherent sheaves on generic analytic  $K3$  surfaces and state that any coherent sheaf admits a locally free resolution of finite length. Finally, Chern classes for coherent sheaves on smooth complex varieties are defined according to [65].

In the last section we prove that a smooth complex variety with the same derived category as a generic analytic  $K3$  surface is a surface of the same type and state Szendrői's conjecture for any (projective or not)  $K3$  surface. We discuss the notion of stability condition due to Bridgeland and use it to fully describe the group of autoequivalences of generic analytic  $K3$  surfaces.

A few warnings are in order at this point. First of all, there are no new results in this presentation and we have deliberately omitted many nice results concerning algebraic  $K3$  surfaces. For example, the section about automorphisms of algebraic  $K3$  surfaces is mainly based on simple examples and none of the beautiful deep results in the literature are considered. This is justified since the algebraic setting is out of the scope of this paper. Secondly, the proofs are described in detail only when the length and the complexity are reasonable. When we feel that it may be worthwhile to give the reader an impression of the techniques used in a given proof which is otherwise too involved, we simply outline the main steps. The final warning concerns the fact that the reader is supposed to have some familiarity with the geometry of  $K3$  surfaces and with the language of derived categories. Although some introductory sections are present in this paper, the less prepared reader is encouraged to have a look at [1,4] and at the first few chapters of [43].

## 2. The geometry of $K3$ surfaces

In this section we discuss a few basic geometric properties of algebraic and generic analytic  $K3$  surfaces. After describing some simple examples, we will introduce the Hodge and lattice structures defined on the second and on the total cohomology groups of any  $K3$  surface. The high points of the theory are certainly the Torelli Theorem and the theorem on the surjectivity of the period map.

While it is very easy to describe explicit examples of algebraic  $K3$  surfaces, the non-algebraic ones are usually presented with a rather abstract approach (e.g. invoking the surjectivity of the period map). At the end of this section, we present beautiful examples of complex non-algebraic surfaces that were first described by McMullen.

For the sake of completeness we include a short section where we collect well-known results in lattice theory.

### 2.1. Lattices

Lattice theory plays a special role in the study of the geometry of  $K3$  surfaces. As we will see in a while, interesting aspects of the geometry of  $K3$  surfaces are encoded by the Hodge and lattice structures defined on their cohomology groups. Due to a celebrated result of Orlov (see Theorem 5.2) the lattice theoretic properties of the cohomology of a  $K3$  surface are also important to detect equivalences between derived categories.

Therefore, not surprisingly, in this paper we will often need a few algebraic results in lattice theory. We list them in this section for the convenience of the reader. The proofs can be found in [64] (see also [23] for a readable account).

First of all, recall that a *lattice* is a free abelian group  $L$  of finite rank with a non-degenerate symmetric bilinear form  $b : L \times L \rightarrow \mathbb{Z}$ . For simplicity, we always write  $x \cdot y := b(x, y)$  and we denote by  $O(L)$  the group of isometries of a lattice  $L$  (i.e. the automorphisms of  $L$  preserving the bilinear form).

**Example 2.1.** (i) The *hyperbolic lattice*  $U$  is the free abelian group  $\mathbb{Z} \oplus \mathbb{Z}$  endowed with the bilinear form represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

with respect to the standard basis.

(ii) Consider the free abelian group  $\mathbb{Z}^{\oplus 8}$  with the bilinear form

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Such a lattice is denoted by  $E_8$ .

(iii) We denote by  $\langle d \rangle$  the lattice  $L = (\mathbb{Z}, b)$  where  $b(h, h) = d$  for a generator  $h$  of the underlying rank one abelian group.

(iv) Given a lattice  $L$  with bilinear form  $b_L$  and an integer  $m \in \mathbb{Z}$ , the lattice  $L(m)$  coincides with  $L$  as a group but its bilinear form  $b_{L(m)}$  is such that  $b_{L(m)}(l_1, l_2) = mb_L(l_1, l_2)$ , for any  $l_1, l_2 \in L$ .

A lattice  $(L, b)$  is *even* if, for all  $x \in L$ ,  $x^2 := b(x, x) \in 2\mathbb{Z}$ , while it is *odd* if there exists  $x \in L$  such that  $b(x, x) \notin 2\mathbb{Z}$ . Given an integral basis for  $L$ , we associate to the bilinear form  $b_L$  a symmetric  $(\text{rk } L \times \text{rk } L)$ -matrix  $S_L$ . The *discriminant* of  $L$  is the integer  $\text{disc}(L) := \det S_L$ . A lattice is *unimodular* if  $\text{disc}(L) = \pm 1$  (notice that  $\text{disc}(L) \neq 0$ , because  $L$  is non-degenerate). Moreover, a lattice  $L$  is *positive definite* if  $x^2 > 0$ , for any  $0 \neq x \in L$  while  $L$  is *negative definite* if  $x^2 < 0$ , for any  $0 \neq x \in L$ . It is *indefinite* if it is neither positive definite nor negative definite. (Notice that  $L(-1)$  is negative definite if and only if  $L$  is positive definite.)

Unimodular indefinite lattices are particularly relevant for the study of  $K3$  surfaces and they are completely classified by the following result.

**Theorem 2.2** ([57]). *Let  $L$  be a unimodular indefinite lattice. If  $L$  is odd then  $L \cong \langle 1 \rangle^{\oplus m} \oplus \langle -1 \rangle^{\oplus n}$ , while if  $L$  is even  $L \cong U^{\oplus m} \oplus E_8(\pm 1)^{\oplus n}$ , for a unique pair of positive integers  $m$  and  $n$ .*

We can associate a finite group to  $L$ . Indeed, consider

$$L^\vee := \text{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) \cong \{l \in L \otimes \mathbb{Q} : l \cdot p \in \mathbb{Z}, \text{ for any } p \in L\}$$

and the natural inclusion  $L \hookrightarrow L^\vee$ . The bilinear form on  $L$  induces a symmetric bilinear form  $b^\vee : L^\vee \times L^\vee \rightarrow \mathbb{Q}$ . The *discriminant group* of  $L$  is the quotient

$$A_L := L^\vee / L.$$

The order of  $A_L$  is  $|\text{disc}(L)|$  (see [4, Lemma 2.1]). Moreover,  $b^\vee$  induces a symmetric bilinear form  $b_L : A_L \times A_L \rightarrow \mathbb{Q}/\mathbb{Z}$  and thus a quadratic form

$$q_L : A_L \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Whenever  $L$  is even,  $q_L$  takes values in  $\mathbb{Q}/2\mathbb{Z}$ . By  $O(A_L)$  we denote the group of automorphisms of  $A_L$  preserving  $q_L$ . We denote by  $\ell(L)$  the minimal number of generators of  $A_L$ . The inclusion of  $L$  into  $L^\vee$  yields a homomorphism  $\Psi : O(L) \rightarrow O(A_L)$ .

The discriminant group of a lattice  $L$  and the group of its automorphisms preserving the quadratic form just defined always play a role when we want to extend isometries of sublattices to isometry of  $L$  itself. Since at some point we will have to do this, we briefly recall the main results proved by Nikulin in this direction.

**Proposition 2.3** ([64], Theorem 1.14.2). *Let  $L$  be an even indefinite lattice such that  $\text{rk } L \geq \ell(L) + 2$ . Then the map  $\Psi : O(L) \rightarrow O(A_L)$  is onto.*

An embedding  $i : V \hookrightarrow L$  of a lattice  $V$  into a lattice  $L$  is *primitive* if  $L/i(V)$  is a free abelian group. One can show that given a primitive sublattice  $K$  of a unimodular lattice  $L$ , there exists a morphism of groups  $\gamma : A_K \rightarrow A_{K^\perp}$ . Moreover,

**Proposition 2.4** ([64], Corollary 1.6.2). *If  $K$  and  $L$  are as above, then the morphism  $\gamma$  induces an isometry  $\gamma : (A_K, q_K) \rightarrow (A_{K^\perp}, -q_{K^\perp})$ .*

Hence,  $\gamma$  defines an isomorphism  $\psi : O(A_K) \rightarrow O(A_{K^\perp})$ . Furthermore, given an even unimodular lattice  $L$ , there are two homomorphisms  $\Psi_1 : O(K) \rightarrow O(A_K)$  and  $\Psi_2 : O(K^\perp) \rightarrow O(A_{K^\perp})$  naturally associated with a primitive sublattice  $K$  of  $L$ .

**Proposition 2.5** ([64], Theorem 1.6.1 and Corollary 1.5.2). *An isometry  $f \in O(K)$  lifts to an isometry in  $O(L)$  if and only if  $\psi(\Psi_1(f)) \in \text{im}(\Psi_2)$ .*

For later use we include here a simple example:

**Example 2.6.** Consider a primitive sublattice  $L = \langle 2d \rangle$  of  $\Lambda$ , with  $d > 0$ . From the definition we have that  $A_L \cong \mathbb{Z}/2d\mathbb{Z}$ . If  $d \neq 1$ , then it was proved in [76] that  $O(A_L)$  has order greater than or equal to 2. In particular, keeping the previous notations,  $\Psi_1(\text{id}) \neq \Psi_1(-\text{id})$ .

### 2.2. K3 surfaces

An analytic K3 surface (or simply a K3 surface) is a complex smooth surface  $X$  with trivial canonical bundle and such that  $H^1(X, \mathbb{Z}) = 0$ . Note that Siu proved in [81] that these properties automatically imply that  $X$  carries a Kähler structure. Although in this paper we will be mainly interested in complex non-projective K3 surfaces let us start with some easy examples of algebraic K3 surfaces.

**Example 2.7** (Complete Intersections). Let  $X$  be a smooth surface in  $\mathbb{P}^{n+2}$  which is the complete intersection of  $n$  hypersurfaces of degrees  $d_1, \dots, d_n$ , with  $d_i > 1$  for  $i \in \{1, \dots, n\}$ . The adjunction formula shows that the dualizing sheaf  $\omega_X$  of  $X$  is  $\mathcal{O}_X((d_1 + \dots + d_n) - (n + 3))$ . Hence  $X$  is a K3 surface if and only if  $d_1 + \dots + d_n = n + 3$ . In particular the only possibilities are:

$$\begin{aligned} n = 1 & \quad d_1 = 4; \\ n = 2 & \quad d_1 = 2 \quad d_2 = 3; \\ n = 3 & \quad d_1 = 2 \quad d_2 = 2 \quad d_3 = 2. \end{aligned}$$

A result of Kodaira proves that K3 surfaces, considered as differentiable manifolds, are all diffeomorphic. In particular, Example 2.7 shows that any K3 surface is diffeomorphic to a smooth quartic surface in  $\mathbb{P}^3$ . Since a surface of this type is simply-connected, any K3 surface is simply-connected as well.

The Néron–Severi group of a K3 surface  $X$  is the free abelian group

$$\text{NS}(X) := c_1(\text{Pic}(X)) \subset H^2(X, \mathbb{Z}),$$

where  $c_1$  is the natural map

$$H^1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow \text{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z})$$

induced by the exponential exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0.$$

By the Hodge decomposition theorem,  $H^1(X, \mathcal{O}_X) = 0$ . Therefore  $c_1$  is injective and  $\text{Pic}(X) \cong \text{NS}(X)$ . The rank  $\rho(X)$  of the free abelian group  $\text{NS}(X)$  is the Picard number of  $X$ .

The orthogonal complement  $T(X) := \text{NS}(X)^\perp \subset H^2(X, \mathbb{Z})$  with respect to the cup product is the transcendental lattice of  $X$ . Notice that if  $X$  is algebraic, the Hodge Index Theorem implies that  $\text{NS}(X)$  has signature  $(1, \rho(X) - 1)$ . In the non-algebraic case this is no longer true. Indeed, we can consider the following example:

**Example 2.8** (Kummer Surfaces). Let  $T$  be a complex torus of dimension 2 and consider the quotient  $S := T/\langle \iota \rangle$ , where  $\iota(a) = -a$  for any  $a \in T$ . The surface  $S$  has 16 singular points corresponding to the points fixed by  $\iota$ . Let  $\tilde{T}$

be the blow-up of  $T$  along those 16 points and let  $\tilde{\iota}$  be the natural extension of  $\iota$  to  $\tilde{T}$  (i.e. the extension of  $\iota$  acting trivially on the points of the exceptional curves). The quotient

$$\text{Km}(T) := \tilde{T}/\langle \tilde{\iota} \rangle$$

is a K3 surface and it is called the *Kummer surface of  $T$* . Observe that  $\text{Km}(T)$  is the crepant resolution of the singular quotient  $T/\langle \iota \rangle$  and  $\text{Km}(T)$  is algebraic if and only if  $T$  is an abelian surface. Moreover, if the signature of  $\text{NS}(T)$  is  $(s, r)$ , then the signature of  $\text{NS}(\text{Km}(T))$  is  $(s, r + 16)$ . For example, the signature  $(0, 16)$  is realized considering a two-dimensional complex torus  $T$  with  $\text{Pic}(T) = 0$ .

The second cohomology group  $H^2(X, \mathbb{Z})$  endowed with the cup product  $(-, -)$  is an even unimodular lattice of signature  $(3, 19)$  (and hence of rank 22). More precisely, due to [Theorem 2.2](#), there exists an isometry

$$H^2(X, \mathbb{Z}) \cong \Lambda := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2},$$

where  $U$  and  $E_8$  are defined in [Example 2.1](#). The lattice  $\Lambda$  is usually called the *K3 lattice*. Notice that due to this description of  $H^2(X, \mathbb{Z})$ ,  $T(X)$  has signature  $(2, 20 - \rho(X))$  if  $X$  is algebraic while it is  $(3, 19)$  if  $X$  is a *generic analytic K3 surface* (i.e.  $\text{Pic}(X) = 0$ ).

Given a K3 surface  $X$ , a *marking* for  $X$  is the choice of an isometry  $\varphi : H^2(X, \mathbb{Z}) \xrightarrow{\sim} \Lambda$ . A *marked K3 surface* is a pair  $(X, \varphi)$ , where  $X$  is a K3 surface and  $\varphi$  is a marking for  $X$ . Two marked K3 surfaces  $(X, \varphi)$  and  $(X', \varphi')$  are isomorphic if there exists an isomorphism  $f : X \xrightarrow{\sim} X'$  such that  $\varphi' = \varphi \circ f^*$ .

### 2.3. Hodge structures

Let  $X$  be a K3 surface. The usual weight-two Hodge structure on  $H^2(X, \mathbb{Z})$  is the decomposition

$$H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X).$$

Since  $X$  has trivial dualizing sheaf,  $H^{2,0}(X) = \langle \sigma_X \rangle \cong \mathbb{C}$ , where

$$(\sigma_X, \sigma_X) = 0 \quad \text{and} \quad (\sigma_X, \overline{\sigma_X}) > 0. \tag{2.1}$$

Let  $X$  and  $Y$  be K3 surfaces. An isometry (with respect to the cup product)  $f : H^2(X, \mathbb{Z}) \xrightarrow{\sim} H^2(Y, \mathbb{Z})$  is a *Hodge isometry* if  $f(H^{2,0}(X)) = H^{2,0}(Y)$ . An isometry  $g : T(X) \xrightarrow{\sim} T(Y)$  is a Hodge isometry if  $g(H^{2,0}(X)) = H^{2,0}(Y)$  (notice that  $H^{2,0}(X) \subseteq T(X) \otimes \mathbb{C}$  and  $H^{2,0}(Y) \subseteq T(Y) \otimes \mathbb{C}$  because  $\text{NS}(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Z})$  and  $\text{NS}(Y) = H^{1,1}(Y) \cap H^2(Y, \mathbb{Z})$ ).

**Remark 2.9.** Notice that the Lefschetz  $(1, 1)$ -theorem implies that the transcendental lattice  $T(X)$  of a K3 surface  $X$  can be described as the minimal primitive sublattice of  $H^2(X, \mathbb{Z})$  such that  $\sigma_X \in T(X) \otimes \mathbb{C}$ . In particular, if  $X$  is generic analytic, then  $T(X) = H^2(X, \mathbb{Z})$ .

The second cohomology group of a K3 surface contains two interesting cones. Indeed, the *positive cone*  $\mathcal{C}_X \subset H^{1,1}(X) \cap H^2(X, \mathbb{R})$  is the connected component of the cone  $\{x \in H^{1,1}(X) \cap H^2(X, \mathbb{R}) : x \cdot x > 0\}$  containing a Kähler class. We denote by  $\mathcal{K}_X$  the *Kähler cone* of  $X$  (i.e. the cone generated by the Kähler classes).

**Remark 2.10.** One can prove that  $\mathcal{K}_X = \{x \in \mathcal{C}_X : (x, \delta) > 0 : \delta \in \Delta(X)\}$ , where  $\Delta(X) = \{y \in \text{NS}(X) : (y, y) = -2\}$ . It is clear that if  $X$  is such that  $\text{NS}(X) = 0$ , then  $\mathcal{K}_X = \mathcal{C}_X$ .

Of course, the Hodge isometry induced on cohomology by any isomorphism  $f : X \xrightarrow{\sim} X$  preserves the Kähler cone. Even more is true. Indeed the Torelli Theorem shows that the converse of this statement holds. Namely the Hodge and lattice structures on the second cohomology group identify a K3 surface up to isomorphism.

**Theorem 2.11 (Strong Torelli Theorem).** *Let  $X$  and  $Y$  be K3 surfaces. Suppose that there exists a Hodge isometry  $g : H^2(X, \mathbb{Z}) \xrightarrow{\sim} H^2(Y, \mathbb{Z})$  which maps a Kähler class on  $X$  into the Kähler cone of  $Y$ . Then there exists a unique isomorphism  $f : X \xrightarrow{\sim} Y$  such that  $f_* = g$ .*

Many people have given different proofs of the previous result at different levels of generality. Among them we recall [[19,55,75](#)] (see also [[1,4](#)]). The following is a weaker form of the previous result:

**Theorem 2.12 (Torelli Theorem).** *Let  $X$  and  $Y$  be  $K3$  surfaces. Then  $X$  and  $Y$  are isomorphic if and only if there exists a Hodge isometry  $H^2(X, \mathbb{Z}) \cong H^2(Y, \mathbb{Z})$ .*

There is a second way to use the weight-two Hodge structure on  $H^2$  to study the geometry of a  $K3$  surface  $X$ . Consider the quasi-projective variety

$$Q := \{\sigma \in \mathbb{P}(\Lambda \otimes \mathbb{C}) : \sigma \cdot \sigma = 0 \text{ and } \sigma \cdot \bar{\sigma} > 0\}$$

which is called the *period domain*. If  $(X, \varphi)$  is a marked  $K3$  surface, (2.1) implies  $[\varphi(\mathbb{C}\sigma_X)] \in Q$  (here  $[\varphi(\mathbb{C}\sigma_X)]$  is the class in  $\mathbb{P}(\Lambda \otimes \mathbb{C})$  of the line  $\varphi(\mathbb{C}\sigma_X) \subset \Lambda \otimes \mathbb{C}$ ).

Let  $\mathcal{M}$  be the moduli space of marked  $K3$  surfaces (see [1,24] for a discussion about the construction of  $\mathcal{M}$ ). The map

$$\theta : \mathcal{M} \longrightarrow Q$$

sending  $(X, \varphi)$  to  $[\varphi(\mathbb{C}\sigma_X)]$  is the *period map* and  $[\varphi(\mathbb{C}\sigma_X)]$  is usually called *period* of the  $K3$  surface  $X$ .

**Theorem 2.13 (Surjectivity of the Period Map).** *The map  $\theta$  is surjective.*

The interested reader can find a readable account of the proof of this deep result in [4].

Consider a marked  $K3$  surface  $(X, \varphi)$  such that  $\text{NS}(X) \neq 0$  and let  $H^{2,0}(X) = \langle \sigma_X \rangle$ . In particular,  $\text{rk } T(X) < 22$  and there exists a primitive sublattice  $L \hookrightarrow \Lambda$  with  $\text{rk } L < 22$  and such that  $[\varphi(\mathbb{C}\sigma_X)] \in Q \cap \mathbb{P}(L \otimes \mathbb{C})$ . Since there exist at most countably many primitive sublattices  $L$  of  $\Lambda$  with rank strictly smaller than 22, the periods of marked  $K3$  surfaces  $(X, \varphi)$  such that  $\text{NS}(X) = 0$  are contained in the complement of the union of countably many hyperplane sections of the period domain  $Q$ . Therefore, a  $K3$  surface  $X$  with  $\text{NS}(X) = 0$  is *generic*.

Passing from  $H^2(X, \mathbb{Z})$  to the total cohomology group  $H^*(X, \mathbb{Z})$  of a  $K3$  surface  $X$ , we see that  $H^*(X, \mathbb{Z})$  has a natural lattice structure induced by the *Mukai pairing* defined by

$$(\alpha, \beta) := -(\alpha_0, \beta_4) + (\alpha_2, \beta_2) - (\alpha_4, \beta_0), \tag{2.2}$$

for every  $\alpha = (\alpha_0, \alpha_2, \alpha_4)$  and  $\beta = (\beta_0, \beta_2, \beta_4)$  in  $H^*(X, \mathbb{Z})$ . The group  $H^*(X, \mathbb{Z})$  endowed with the Mukai pairing is called *Mukai lattice* and we write  $\tilde{H}(X, \mathbb{Z})$  for it. Since  $H^0(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$  with the cup product is isometric to  $U$ , we have a natural isometry of lattices

$$\tilde{H}(X, \mathbb{Z}) \cong \tilde{\Lambda} := \Lambda \oplus U.$$

We can now define a weight-two Hodge structure on the total cohomology group  $H^*(X, \mathbb{Z})$  in the following natural way (see [39,42,44] for a more general setting):

$$\begin{aligned} \tilde{H}^{2,0}(X) &:= H^{2,0}(X), \\ \tilde{H}^{0,2}(X) &:= H^{0,2}(X), \\ \tilde{H}^{1,1}(X) &:= H^0(X, \mathbb{C}) \oplus H^{1,1}(X) \oplus H^4(X, \mathbb{C}). \end{aligned} \tag{2.3}$$

An isomorphism  $f : \tilde{H}(X, \mathbb{Z}) \xrightarrow{\sim} \tilde{H}(Y, \mathbb{Z})$  is a Hodge isometry if it is an isometry of lattices which preserves the Hodge structures just defined.

As in the case of the second cohomology group, for a  $K3$  surface  $X$ , the lattice  $\tilde{H}(X, \mathbb{Z})$  has some very interesting substructure. Indeed, let  $\sigma$  be a generator of  $H^{2,0}(X)$  and  $\omega$  a Kähler class (e.g. an ample class if  $X$  is algebraic), then

$$P(X, \sigma, \omega) := \langle \text{Re}(\sigma), \text{Im}(\sigma), 1 - \omega^2/2, \omega \rangle, \tag{2.4}$$

is a positive four-space in  $\tilde{H}(X, \mathbb{R})$  (here  $\text{Re}(\sigma)$  and  $\text{Im}(\sigma)$  are the real and imaginary parts of  $\sigma$ ). It comes, by the choice of the basis, with a natural orientation. It is easy to see that this orientation is independent of the choice of  $\sigma$  and  $\omega$ .

In general, let  $g : \Gamma \rightarrow \Gamma'$  be an isometry of lattices with signature  $(4, t)$ . Suppose that positive four-spaces  $V \subset \Gamma_{\mathbb{R}} := \Gamma \otimes \mathbb{R}$  and  $V' \subset \Gamma'_{\mathbb{R}} := \Gamma' \otimes \mathbb{R}$  and orientations for both of them have been chosen. Then one says that the isometry  $g$  *preserves the given orientation of the positive directions* (or, simply, that  $g$  is *orientation preserving*) if the composition of

$$g_{\mathbb{R}} : V \longrightarrow \Gamma'_{\mathbb{R}}$$



with the orthogonal projection  $\Gamma'_{\mathbb{R}} \rightarrow V'$  is compatible with the given orientations of  $V$  and  $V'$ . By  $O_+(\Gamma)$  one denotes the group of all orientation preserving isometries of  $\Gamma$ .

A Hodge isometry  $f : \tilde{H}(X, \mathbb{Z}) \xrightarrow{\sim} \tilde{H}(Y, \mathbb{Z})$  is *orientation preserving* if it preserves the orientation of the four positive directions previously defined.

**Example 2.14.** (i) It is easy to see that, for any  $K3$  surface  $X$ , the Hodge isometry  $j := \text{id}_{H^2(X, \mathbb{Z})} \oplus (-\text{id})_{H^0(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})}$  is not orientation preserving.

(ii) Consider now  $\delta = (1, 0, 1) \in \tilde{H}(X, \mathbb{Z})$  and take the Hodge isometry defined by

$$s_{\delta}(x) = x - 2 \frac{\langle \delta, x \rangle}{\langle \delta, \delta \rangle} \delta = x + \langle \delta, x \rangle \delta,$$

for any  $x \in \tilde{H}(X, \mathbb{Z})$ . Clearly  $s_{\delta}$  is a reflection and is orientation preserving.

(iii) Let  $f : X \xrightarrow{\sim} Y$  be an isomorphism of  $K3$  surfaces and take the natural identifications  $H^0(X, \mathbb{Z}) = H^0(Y, \mathbb{Z}) = \mathbb{Z}$  and  $H^4(X, \mathbb{Z}) = H^4(Y, \mathbb{Z}) = \mathbb{Z}$ . The Hodge isometry  $f^*|_{H^2(Y, \mathbb{Z})} \oplus (\text{id})_{\mathbb{Z} \oplus \mathbb{Z}} : \tilde{H}(Y, \mathbb{Z}) \xrightarrow{\sim} \tilde{H}(X, \mathbb{Z})$  preserves the orientation of the four positive directions.

### 2.4. McMullen's examples

In this section we describe some generic analytic  $K3$  surfaces following McMullen's original idea in [58] (see also [37, Thm. 1.7]). This beautiful construction requires some preliminary results about Salem numbers and Salem polynomials (mainly contained in [37]) which we briefly summarize.

**Definition 2.15.** (i) An irreducible monic polynomial  $S(x) \in \mathbb{Z}[x]$  is a *Salem polynomial* if the set of its roots is  $\{\lambda, \lambda^{-1}, a_1, \overline{a_1}, \dots, a_n, \overline{a_n}\}$ , where  $\lambda \in \mathbb{R}, \lambda > 1$  and  $|a_i| = 1$ , for any  $i \in \{1, \dots, n\}$ .

(ii) A Salem polynomial  $S(x)$  is *unramified* if  $|S(-1)| = |S(1)| = 1$ .

(iii) An algebraic real number  $\lambda > 1$  is a *Salem number* if the minimal irreducible monic polynomial  $P(x)$  such that  $P(\lambda) = 0$  is a Salem polynomial.

By definition a Salem polynomial has even degree. Let  $n \geq 3$  be an odd integer and consider a separable polynomial  $C(x) \in \mathbb{Z}[x]$  of degree  $n - 3$  and whose roots are contained in the interval  $[-2, 2]$  (recall that  $C(x)$  is separable if it does not have multiple roots in  $\mathbb{C}$ ). Take now the polynomial

$$R(x) = C(x)(x^2 - 4)(x - a) - 1,$$

where  $a \in \mathbb{Z}$ . Finally, consider the polynomial  $S(x) = x^n R(x + x^{-1})$  which has degree  $2n$ . In [37] Gross and McMullen showed that  $S(x)$  is an unramified Salem polynomial for any  $a \gg 0$ . Hence the following holds true:

**Theorem 2.16** ([37], Theorem 1.6). *For any odd integer  $n \geq 3$  there are infinitely many unramified Salem polynomials of degree  $2n$ .*

An interesting property of Salem polynomials contained in [58, Sect. 10] and which will be used in Section 3.1 is given by the following lemma.

**Lemma 2.17.** *Given a positive integer  $n$  and two real numbers  $c_1 < c_2$ , there exist finitely many Salem polynomials  $S(x)$  such that  $\text{deg } S(x) = 2n$  and  $c_1 < \text{tr } S(x) < c_2$ , where  $\text{tr } S(x)$  is the sum of the roots of  $S(x)$ .*

**Proof.** By definition, the set of roots of a Salem polynomial of degree  $2n$  is of the form

$$\{\lambda, \lambda^{-1}, a_1, \overline{a_1}, \dots, a_{n-1}, \overline{a_{n-1}}\},$$

where  $\lambda \in \mathbb{R}, \lambda > 1$  and  $|a_i| = 1$ , for any  $i \in \{1, \dots, n - 1\}$ . In particular,

$$1 - n \leq \sum_{i=1}^{n-1} a_i + \overline{a_i} \leq n - 1$$

and, by assumption,

$$c_1 \leq \lambda + \lambda^{-1} + \sum_{i=1}^{n-1} a_i + \bar{a}_i \leq c_2.$$

Hence  $\lambda$  is bounded from above and all the roots of  $S(x)$  are contained in a compact subset of  $\mathbb{C}$ .

Let us write  $S(x) = x^{2n} + \sum_{k=1}^{2n-1} b_k x^k + 1$ , where  $b_k$  is an integer which is a symmetric function of the roots of  $S(x)$ . Therefore, the coefficients of  $S(x)$  are contained in a compact subset of  $\mathbb{R}$ . This concludes the proof.  $\square$

The main idea in McMullen’s construction is to define a  $K3$  surface  $X$  with an automorphism  $f \in \text{Aut}(X)$  such that the minimal polynomial of the induced isometry  $f^* : H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z})$  is a Salem polynomial.

As a first step in this direction we consider an unramified Salem polynomial  $S(x)$  of degree  $22 = \text{rk } \Lambda$  whose existence is granted by Theorem 2.16. Let  $\lambda$  be the unique root of  $S(x)$  such that  $\lambda \in \mathbb{R}$  and  $\lambda > 1$ . Consider the ring

$$B := \mathbb{Z}[x]/(S(x))$$

and let  $K$  be its field of fractions.

The ring  $B$  contains a subring  $R$  generated by the element  $y = x + x^{-1}$  (see [58, Sect. 8]). Choosing a unit  $u \in R$ , we define on  $B$  the following bilinear symmetric form:

$$\langle f_1, f_2 \rangle_{B,u} := \text{Tr}_{\mathbb{Q}}^K \left( \frac{u f_1(x) f_2(x^{-1})}{r'(y)} \right), \tag{2.5}$$

where  $r$  is the minimal polynomial of  $\lambda + \lambda^{-1}$  and  $r'$  is its derivative. McMullen showed that  $\langle f_1, f_2 \rangle_{B,u} \in \mathbb{Z}$ , for any  $f_1, f_2 \in B$ , and so the pair  $(B, \langle -, - \rangle_{B,u})$  is a lattice.

The choice of the unit  $u$  is extremely relevant in McMullen’s construction. In fact, in [58, Sect. 8] he proved the following result:

**Proposition 2.18.** *There exists  $u \in R$  such that the lattice  $(B, \langle -, - \rangle_{B,u})$  is isometric to the  $K3$  lattice  $\Lambda$ .*

In particular, the lattice  $(B, \langle -, - \rangle_{B,u})$  has the same lattice structure as the second cohomology group of any  $K3$  surface.

The second key step consists in defining an appropriate weight-two Hodge structure on  $B$ . To this end, consider now the isomorphism  $F : B \rightarrow B$  such that  $F(g) = xg$ . By definition  $F$  preserves the bilinear form (2.5) and hence yields an isometry of the lattice  $(B, \langle -, - \rangle_{B,u})$ . McMullen observes that the morphism  $F + F^{-1} : B \otimes \mathbb{C} \rightarrow B \otimes \mathbb{C}$  has a unique eigenspace  $E$  whose signature is  $(2, 0)$ . Moreover  $E$  can be decomposed as the direct sum  $E = H_1 \oplus H_2$  where  $H_1$  is the eigenspace of an eigenvalue  $\delta$  of  $F$ , while  $H_2$  is the eigenspace of the eigenvalue  $\delta^{-1}$ . In particular  $E$  is invariant under the action of the isometry  $F$ . Therefore, the lattice  $(B, \langle -, - \rangle_{B,u})$  has a natural weight-two Hodge structure defined by

$$(B \otimes \mathbb{C})^{2,0} := H_1 \quad (B \otimes \mathbb{C})^{0,2} := H_2 \quad (B \otimes \mathbb{C})^{1,1} := (H_1)^{\perp \langle \cdot, \cdot \rangle_{B,u}}. \tag{2.6}$$

Moreover, if  $H_1 = \langle \sigma \rangle$ , then  $H_2 = \langle \bar{\sigma} \rangle$  and

$$\langle \sigma, \sigma \rangle_{B,u} = 0 \quad \langle \sigma, \bar{\sigma} \rangle_{B,u} > 0. \tag{2.7}$$

Since by Theorem 2.13 a  $K3$  surface is determined (up to isomorphism) by the choice of a weight-two Hodge structure as in (2.6) and (2.7), there exists a  $K3$  surface  $X$  and a Hodge isometry  $(H^2(X, \mathbb{Z}), (-, -)) \cong (B, \langle -, - \rangle_{B,u})$ .

Now we can easily prove (using [58, Thm. 3.4]) that the surface  $X$  just constructed is a generic analytic  $K3$  surface.

**Lemma 2.19.** *Let  $X$  be a  $K3$  surface as above. Then  $\text{NS}(X) = 0$ .*

**Proof.** Due to [58, Sect. 8], the minimal polynomial of the isometry  $F : B \xrightarrow{\sim} B$  is the Salem polynomial  $S(x)$ . Suppose  $\text{NS}(X) \neq 0$  and let  $F_{\mathbb{Q}}$  be the  $\mathbb{Q}$ -linear extension of  $F$ . Then  $F_{\mathbb{Q}} = F_1 \times F_2$ , where  $F_1 \in O(\text{NS}(X) \otimes \mathbb{Q})$  and  $F_2 \in O(T(X) \otimes \mathbb{Q})$ . But this is impossible, since  $S(x)$  is irreducible and then  $F$  has no rational invariant subspaces.  $\square$

By abuse of notation, let us denote by  $F \in O(H^2(X, \mathbb{Z}))$  the isometry induced by  $F : B \rightarrow B$ .

**Lemma 2.20.** *There exists  $f \in \text{Aut}(X)$  such that  $f^* = F : H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z})$ .*

**Proof.** Since  $\text{NS}(X) = 0$ , due to Remark 2.10,  $\mathcal{K}_X = \mathcal{C}_X$ . Hence, by Theorem 2.11, it is enough to show that  $F(\mathcal{C}_X) = \mathcal{C}_X$ .

Let  $E(\lambda)$  be the eigenspace of the eigenvalue  $\lambda$  (which is a Salem number) and let  $\alpha \in \mathbb{C}$  be such that  $F(\sigma_X) = \alpha\sigma_X$ , where  $H^{2,0}(X) = \langle \sigma_X \rangle$ . As

$$(\sigma_X, \overline{\sigma_X}) = (F(\sigma_X), F(\overline{\sigma_X})) = |\alpha|^2(\sigma_X, \overline{\sigma_X}) \neq 0,$$

the eigenvalue  $\alpha$  is in the unit circle and  $E(\lambda) \subset H^{1,1}(X) \cap H^2(X, \mathbb{R})$ . Moreover, since  $\lambda$  is real, there exists  $0 \neq v \in E(\lambda) \cap H^2(X, \mathbb{R})$ . As  $F$  is an isometry,  $(v, v) = (F(v), F(v)) = \lambda^2(v, v)$  and then  $(v, v) = 0$ . Hence, up to changing  $v$  with  $-v$  we can suppose  $v$  in the boundary of the positive cone. In particular,  $F$  preserves the closure of the positive cone and, therefore,  $\mathcal{K}_X$ .  $\square$

At the end we produced a  $K3$  surface  $X$  with  $\text{NS}(X) = 0$  and such that  $\text{Aut}(X)$  contains an automorphism  $f$  whose induced isometry  $f^* : H^2(X, \mathbb{Z}) \xrightarrow{\sim} H^2(X, \mathbb{Z})$  has a characteristic polynomial of Salem type.

### 3. Automorphisms

In this section we study the automorphism groups of non-algebraic  $K3$  surfaces with Picard number 0. As it will turn out (Theorem 3.6) such a group is either trivial or isomorphic to  $\mathbb{Z}$ . The latter holds true if and only if the  $K3$  surface is constructed as in Section 2.4. Section 3.2 is devoted to showing a few differences with the algebraic case.

We will reconsider the automorphisms of a non-projective generic  $K3$  surface under a different light in Section 5. There we will show the close relation between the automorphism group and the group of autoequivalences of the derived category.

#### 3.1. The non-algebraic generic case

Since any automorphism  $f : X \xrightarrow{\sim} X$  of a  $K3$  surface  $X$  induces a Hodge isometry  $f^* : H^2(X, \mathbb{Z}) \xrightarrow{\sim} H^2(X, \mathbb{Z})$ , there exists  $\chi(f) \in \mathbb{C}^\times$  such that  $f^*(\sigma_X) = \chi(f)\sigma_X$ , where  $H^{2,0}(X) = \langle \sigma_X \rangle$ . Hence we get a natural map

$$\chi : \text{Aut}(X) \longrightarrow \mathbb{C}^\times \tag{3.1}$$

sending  $f$  to  $\chi(f)$ .

**Lemma 3.1.** (i) *For any  $f \in \text{Aut}(X)$  of finite order, the complex number  $\chi(f)$  is a root of unity.*

(ii) *If  $\rho(X) = 0$ , then the morphism  $\chi$  is injective.*

**Proof.** To prove (i), suppose that  $f \in \text{Aut}(X)$  has finite order. Then  $\chi(\langle f \rangle)$  is a finite cyclic subgroup of  $\mathbb{C}^\times$ . Hence all the eigenvalues of  $f^*$  are roots of unity (see also [91]).

For (ii), suppose that there exists  $f \in \text{Aut}(X)$  such that  $f^*(\sigma_X) = \sigma_X$ . This means that  $f^*$  has eigenvalue 1 and then the primitive sublattice  $T := \{x \in H^2(X, \mathbb{Z}) : f^*(x) = x\}$  is non-empty. Moreover  $\sigma_X \in T \otimes \mathbb{C}$ . By Remark 2.9,  $T = T(X) = H^2(X, \mathbb{Z})$  and so  $f^* = \text{id}$ . By Theorem 2.11,  $f = \text{id}$ .  $\square$

We now prove that the automorphism group of non-algebraic generic  $K3$  surfaces does not contain non-trivial automorphisms of finite order.

**Proposition 3.2** ([63], Theorem 3.1). *Let  $X$  be a  $K3$  surface such that  $\rho(X) = 0$  and let  $f \in \text{Aut}(X)$  be of finite order. Then  $f = \text{id}$ .*

**Proof.** Due to Lemma 3.1, it is enough to prove that if  $f$  has finite order then  $\chi(f) = 1$ . Suppose  $\chi(f) \neq 1$  and consider the quotient  $X' := X/\langle f \rangle$ . Since  $\langle f \rangle$  is finite,  $X'$  has a finite number of singular points. Let  $g : Y \rightarrow X'$  be a resolution of these singularities. Of course, since  $X$  is Kähler,  $Y$  is Kähler.

Let  $\sigma$  be a holomorphic 2-form on  $Y$ . We can lift  $\sigma$  to a meromorphic 2-form  $\sigma'$  on  $X$  via the natural quotient map  $X \rightarrow X'$ . The form  $\sigma'$  is not defined just in a finite number of points. Hence it can be extended to a holomorphic 2-form  $\sigma''$  on  $X$  which, moreover, must be invariant under the action of  $\langle f \rangle$ . Since we are supposing  $\chi(f) \neq 1$ ,  $\sigma''$  must be trivial and  $\sigma$  must be trivial as well. In particular  $\dim H^{2,0}(Y) = 0$  and  $Y$  is a smooth algebraic surface (indeed,  $\text{NS}(Y)$  must contain  $h$  such that  $h^2 > 0$ ). This would imply that  $\text{NS}(X) \neq 0$  which contradicts our assumption.  $\square$

Although the group  $\text{Aut}(X)$  cannot contain automorphisms of finite order if  $\rho(X) = 0$ , there may be  $f \in \text{Aut}(X)$  with infinite order. McMullen’s construction in Section 2.4 provides examples of this type. Indeed, if  $X$  is a K3 surface as in Section 2.4, Lemma 2.20 shows that there exists  $f \in \text{Aut}(X)$  whose minimal polynomial  $P(x) \in \mathbb{Z}[x]$  is a Salem unramified polynomial. Since  $P(x)$  has a real eigenvalue  $\lambda > 1$ ,  $f$  has infinite order (Proposition 3.2).

**Proposition 3.3** ([58], Theorem 3.2). *Let  $X$  be a K3 surface such that  $\rho(X) = 0$  and let  $f \in \text{Aut}(X)$  be of infinite order. Then the minimal polynomial of  $f^*$  is a Salem polynomial.*

**Proof.** First notice that  $|\chi(f)| \neq 1$ . Indeed, if  $\chi(f)$  is an  $n$ th root of unity, then  $\chi(f^n) = 1$  and by item (ii) of Lemma 3.1  $f^n = \text{id}$ , which is a contradiction.

More precisely,  $f^*$  has an eigenvalue  $\lambda$  such that  $|\lambda| \neq 1$ . Indeed, all the eigenvalues of  $f^*$  are also roots of its characteristic polynomial in  $\mathbb{Z}[x]$ . Since  $\det(f^*) = \pm 1$ , the eigenvalues of  $f^*$  with absolute value 1 would be roots of unity. This contradicts the previous remark.

Let  $\lambda$  be an eigenvalue of  $f^*$  such that  $|\lambda| > 1$  (which always exists since there is an eigenvalue  $\lambda'$  with  $|\lambda'| \neq 1$ ). The pull-back  $f^*$  preserves the subspace  $H^{1,1}(X) \subset H^2(X, \mathbb{C})$ . Hence  $f^* = F_1 \times F_2 \in O(1, 19) \times O(2, 0)$ . Due to [58, Lemma 3.1],  $h \in O(p, q)$  has at most  $\min\{p, q\}$  eigenvalues outside the unit circle. Hence  $f^*$  has at most one eigenvalue  $\lambda$  with  $|\lambda| > 1$ . Notice that the uniqueness also implies that such an eigenvalue is real.

The fact that  $\lambda > 1$  follows easily from the remark that  $f^*$ , being induced by an automorphism, must preserve the cone  $\mathcal{K}_X$  (for more details on this, see the proof of Lemma 2.20).  $\square$

We are now ready to describe the automorphism group of any K3 surface  $X$  with  $\rho(X) = 0$ . In this sense, according to [70], we prove now a weak version of the much stronger result [70, Thm. 1.5].

**Theorem 3.4** ([70], Theorem 1.5). *Let  $X$  be a K3 surface such that  $\rho(X) = 0$ . Then either  $\text{Aut}(X) = \{\text{id}\}$  or  $\text{Aut}(X) = \langle f \rangle \cong \mathbb{Z}$ .*

**Proof.** Due to [70, Prop. 4.3] the group  $\text{Aut}(X)$  is a finitely generated free abelian group. Now suppose that  $\text{id} \neq f, g \in \text{Aut}(X)$  are two distinct generators.

To shorten the notation we put  $F := f^*$  and  $G := g^*$ . Let  $S_F(x), S_G(x), \lambda_F$  and  $\lambda_G$  be the corresponding Salem polynomials and Salem numbers. We already know (Lemma 3.1) that the morphism

$$\chi : \langle f, g \rangle \longrightarrow \mathbb{C}^\times$$

is injective. Therefore, since  $\mathbb{C}^\times$  is commutative,  $FG = GF$ .

For any pair  $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ , we denote by  $S_{(m,n)}(x)$  the Salem polynomial of the isometry  $F^m G^n$ . Without loss of generality we can suppose that the Salem number of  $S_{(m,n)}(x)$  is  $\lambda_F^m \lambda_G^n$  (notice that as a consequence of Proposition 3.3 and of the fact that  $FG = GF$ ,  $F$  and  $G$  are simultaneously diagonalizable).

There exists two real numbers  $1 < c_1 < c_2$  such that the set  $M_1 := \{(m, n) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\} : c_1 < \lambda_F^m \lambda_G^n < c_2\}$  is infinite. Indeed, choose  $c_1$  and  $c_2$  such that  $\lambda_G < c_2/c_1$  (i.e.  $\log(\lambda_G) < \log(c_2) - \log(c_1)$ ). Hence for  $m$  negative there always exists at least a positive integer  $n$  such that

$$\log(c_1) - m \log(\lambda_F) < n \log(\lambda_G) < \log(c_2) - m \log(\lambda_F).$$

This is equivalent to  $c_1 < \lambda_F^m \lambda_G^n < c_2$ .

Due to Lemma 2.17, there exists a Salem polynomial  $S(x)$  such that the set  $M_2 := \{(m, n) \in M_1 : S_{(m,n)}(x) = S(x)\}$  is infinite. Since  $\chi(f^m \circ g^n)$  is a root of  $S(x)$  for any  $(m, n) \in M_2$  and as  $S(x)$  has only finitely many roots, there are  $m_1, n_1 \in \mathbb{Z}$  such that  $\chi(f^{m_1} \circ g^{n_1}) = \chi(f^{m_2} \circ g^{n_2})$ , for any  $(m_2, n_2) \in M_1$ . This means that  $\chi(f^{m_1-m_2} \circ g^{n_1-n_2}) = 1$  and then  $f^{m_1-m_2} \circ g^{n_1-n_2} = \text{id}$  (by Proposition 3.2).

As  $\chi(\langle f, g \rangle)$  is a free abelian group of finite rank [70, Prop. 4.3] and  $f$  and  $g$  are generators,  $F^{m_1-m_2} \circ G^{n_1-n_2}$  is the identity if and only if  $m_1 = m_2$  and  $n_1 = n_2$ . In particular,  $M_2 = \{(m_1, n_1)\}$  which is a contradiction.  $\square$

The following result shows that the case of  $K3$  surfaces  $X$  with  $\text{NS}(X) = 0$  and with an automorphism of infinite order is relatively rare.

**Corollary 3.5** ([70], Proposition 6.2). *There are at most countably many  $K3$  surfaces  $X$  with  $\rho(X) = 0$  and with  $f \in \text{Aut}(X)$  of infinite order.*

**Proof.** Suppose that  $F \in O(\Lambda)$  has infinite order and that its minimal polynomial is a Salem polynomial  $S(x)$  of degree 22. Since  $S(x)$  is irreducible, the eigenvalues of  $F$  are distinct and the corresponding eigenspaces have dimension 1. Due to Lemma 2.17 there are at most countably many Salem polynomials of degree 22. Hence there are only countably many subspaces of dimension 1 of  $\Lambda \otimes \mathbb{C}$  which are eigenspaces of the  $\mathbb{C}$ -linear extension of some  $F \in O(\Lambda)$  of infinite order with minimal polynomial of Salem type.

To conclude the proof observe that if  $f \in \text{Aut}(X)$  has infinite order, then the characteristic polynomial of  $f^*$  is a Salem polynomial of degree 22 and that  $H^{2,0}(X)$  is an eigenspace of dimension 1 of the  $\mathbb{C}$ -linear extension of  $f^*$ .  $\square$

For the sake of simplicity, we summarize the previous results in the following compact form:

**Theorem 3.6.** *Let  $X$  be a  $K3$  surface such that  $\rho(X) = 0$ . Then either  $\text{Aut}(X) = \{\text{id}\}$  or  $\text{Aut}(X) \cong \mathbb{Z}$ . Moreover the latter happens for countably many  $K3$  surfaces.*

### 3.2. The algebraic case

The main difference between the automorphism groups of generic non-algebraic and algebraic  $K3$  surfaces is that in the first case a complete description can be given while in the second one only partial results are available (e.g. [50–53]). This is because of the richer geometry of algebraic  $K3$  surfaces. Although the many partial results about the automorphism group of algebraic  $K3$  surfaces are very interesting, the techniques appearing in their proofs are far beyond the scope of this paper and so we will not consider them here.

More modestly, the aim of this section is to establish a comparison between the algebraic and the non-algebraic settings mainly through a few simple examples.

Given a  $K3$  surface  $X$ , define the group

$$O_{\text{Hodge}}(X) := \{f \in O(T(X)) : f(H^{2,0}(X)) = H^{2,0}(X)\}.$$

The following lemma is essentially [69, Lemma 4.1].

**Lemma 3.7.** *Let  $X$  be an algebraic  $K3$  surface with odd Picard number. Then  $O_{\text{Hodge}}(X) = \{\pm \text{id}\}$ .*

**Proof.** Let  $H^{2,0}(X) = \langle \sigma_X \rangle$ . We can write  $T(X) \otimes \mathbb{R} = P \oplus N$ , where  $P = T(X) \otimes \mathbb{R} \cap (\mathbb{C}\sigma_X \oplus \overline{\mathbb{C}\sigma_X})$ ,  $N = T(X) \otimes \mathbb{R} \cap H^{1,1}(X)$  and  $\sigma_X$  is a generator of  $H^{2,0}(X)$ . By the Hodge Index Theorem,  $N$  is negative definite and any Hodge isometry  $h : T(X) \rightarrow T(X)$  is such that the  $\mathbb{C}$ -linear extension  $h_{\mathbb{C}}$  is in  $O(P) \times O(N)$ . The subspaces  $P$  and  $N$  are definite and thus  $h$  is diagonalizable over  $\mathbb{C}$  with eigenvalues with absolute value 1.

Since  $h$  is defined over  $\mathbb{Z}$ , the eigenvalues are algebraic integers and thus they are roots of unity. In particular, there is a natural number  $n$  such that  $h(\sigma_X) = \zeta_n \sigma_X$ , where  $\zeta_n$  is a primitive  $n$ th root of unity. This implies that the order of  $h$  is  $n$ . Indeed, the primitive sublattice  $T = \{x \in T(X) : h^n(x) = x\}$  is such that  $\sigma_X \in T \otimes \mathbb{C}$ . Remark 2.9 implies  $T = T(X)$ .

Obviously,  $\dim N = \text{rk } T(X) - 2$  is odd. Since  $h$  restricts to  $N$ , there exists a non-trivial  $x \in T(X)$  such that either  $h(x) = x$  or  $h(x) = -x$ . We can show that if  $h$  has eigenvalue 1 then  $h = \text{id}$ . Indeed, if  $h(x) = x$ , then  $(\zeta_n \sigma_X, x) = (h(\sigma_X), h(x)) = (\sigma_X, x)$ . As  $n \neq 1$ ,  $(\sigma_X, x) = 0$  and  $x \in \text{NS}(X)$ . This is a contradiction because  $x \in T(X)$ .

Similarly one shows that if there exists an  $x \in T(X)$  such that  $h(x) = -x$ , then  $h = -\text{id}$ .  $\square$

Applying the previous result we get a complete description of the group of automorphisms of some  $K3$  surfaces with Picard number 1. This shows a nice analogy with the non-algebraic case analyzed in the previous section.

**Proposition 3.8.** *Let  $X$  be a projective  $K3$  surface such that  $\text{NS}(X) = \langle \ell \rangle$  and  $(\ell, \ell) = 2d > 2$ . Then  $\text{Aut}(X) = \{\text{id}\}$ .*

**Proof.** Take  $f \in \text{Aut}(X)$ . By Lemma 3.7,  $F := f^*|_{T(X)} = \pm \text{id}$ . On the other hand  $G := f^*|_{\text{NS}(X)} = \text{id}$  because  $G$  must preserve  $\mathcal{K}_X$ . Since  $F$  extends to  $f^*$ , Proposition 2.5 and Example 2.6 imply  $F = \text{id}$  and  $f^* = \text{id}$ . By the Strong Torelli Theorem (Theorem 2.11),  $f = \text{id}$ .  $\square$

**Example 3.9.** An example of an algebraic  $K3$  surface with Picard number 1 not satisfying the hypotheses in the previous proposition is obtained by considering a smooth plane sextic  $C \subset \mathbb{P}^2$ . The double cover of  $\mathbb{P}^2$  ramified along  $C$  is an algebraic  $K3$  surface. If  $C$  is sufficiently generic, then  $\rho(X) = 1$ . Notice that  $X$  has always an automorphism of order two given by the involution exchanging the two leaves of the covering.

One can consider other algebraic  $K3$  surfaces  $X$  with non-trivial automorphisms of finite order which do not exist in the generic non-projective case. For example, there exist  $K3$  surfaces  $X$  with an involution  $f \in \text{Aut}(X)$  such that  $f^*|_{T(X)} = \text{id}$ . Such an automorphism is usually called *Nikulin involution* (see [32,33,59,61,63]). More generally an automorphism  $f \in \text{Aut}(X)$  of finite order and such that  $f^*|_{T(X)} = \text{id}$  is called *symplectic*. If a  $K3$  surface  $X$  has symplectic automorphisms, then  $\text{rk NS}(X) \geq 9$  [63]. Just to give some precise references, we recall that the case of  $K3$  surfaces  $X$  such that  $\text{Aut}(X)$  contains a finite subgroup  $G$  whose elements are symplectic automorphisms was first studied in [63] for  $G$  abelian and in [61] for  $G$  non-abelian.

To produce examples of this type, take  $\sigma \in (U \oplus U) \otimes \mathbb{C}$  such that

$$(\sigma, \sigma) = 0 \quad \text{and} \quad (\sigma, \bar{\sigma}) > 0.$$

Since there is a primitive embedding  $U \oplus U \hookrightarrow \Lambda$ , the class of the line  $\mathbb{C}\sigma$  lies in  $\mathcal{Q}$  and by the Surjectivity of the Period Map (Theorem 2.13), there exists a marked  $K3$  surface  $(X, \varphi)$  such that  $\varphi(\mathbb{C}\sigma_X) = \mathbb{C}\sigma$ , where  $H^{2,0}(X) = \langle \sigma_X \rangle$ . Notice that, by the choice of  $\sigma$ ,  $T(X) \hookrightarrow U \oplus U$ . In particular, there exists a primitive embedding  $E_8(-1)^{\oplus 2} \hookrightarrow \text{NS}(X)$ . Theorem 5.7 in [59] ensures that  $X$  has a Nikulin involution and hence a non-trivial automorphism of finite order. To be precise the isometry induced on cohomology by this automorphism exchanges the two copies of  $E_8(-1)$  and fixes the rest of the  $K3$  lattice. Other examples of  $K3$  surfaces with automorphisms group isomorphic to  $\mathbb{Z}/2\mathbb{Z}$  are discussed in [30].

Nevertheless, as in the non-algebraic case, the automorphism group of a  $K3$  surface could be infinite. Indeed, for an algebraic  $K3$  surface  $X$  and for any  $\delta \in \Delta(X)$  (see Remark 2.10 for the definition of  $\Delta(X)$ ), define the reflection in  $O(H^2(X, \mathbb{Z}))$

$$r_\delta(x) := x + (x, \delta)\delta,$$

for any  $x \in H^2(X, \mathbb{Z})$  (see [25] for an extensive use of reflections in the study of automorphism groups of  $K3$  surfaces). Let  $W(X)$  be the Weyl group generated by the reflections  $r_\delta$ . Since any  $\delta \in \Delta(X)$  is orthogonal to the transcendental lattice  $T(X)$ ,  $W(X)$  can be also seen as a subgroup of  $O(\text{NS}(X))$ .

**Theorem 3.10** ([75]). *Let  $X$  be an algebraic  $K3$  surface. Then  $\text{Aut}(X)$  is finite if and only if*

$$O(\text{NS}(X))/W(X)$$

*is a finite group.*

**Example 3.11.** (i) Following [90], one can prove that the generic algebraic  $K3$  surface  $X$  which is a hypersurface of bidegree  $(1, 1)$  in  $\mathbb{P}^2 \times \mathbb{P}^2$  has automorphism group isomorphic to the infinite dihedral group. Other examples of projective  $K3$  surfaces with infinite (non-cyclic) automorphisms group are provided in [31] where the case of the double cover of  $\mathbb{P}^2$  is considered (see Example 3.9).

(ii) One can also produce  $K3$  surfaces  $X$  such that  $\text{Aut}(X) \cong \mathbb{Z}$ . Indeed, following [7], this is the case if  $X$  is a generic projective  $K3$  surface such that the intersection form on  $\text{NS}(X)$  is represented by a diagonal matrix

$$\begin{pmatrix} 2nd & 0 \\ 0 & -2n \end{pmatrix},$$

where  $n \geq 2$  and  $d$  is not a square. The proof of this fact involves a direct calculation of the solutions of some Pell equations.

#### 4. Coherent sheaves and derived categories

In this section we move towards the second main topic of this paper: the description of the group of autoequivalences of the derived category of a generic analytic  $K3$  surface.

As a first step we discuss a few properties of the abelian category of coherent sheaves on surfaces of this type. Not surprisingly, this will lead to another key difference with respect to the algebraic case. Indeed an interesting result of Verbitsky (Theorem 4.2) shows that two smooth analytic  $K3$  surfaces with trivial Néron–Severi lattices have equivalent abelian categories of coherent sheaves. Notice that in the algebraic case this does not happen due to a well-known result of Gabriel (Theorem 4.1). As a consequence of Verbitsky’s result, not all equivalences are of Fourier–Mukai type (Proposition 4.22). Hence the celebrated result of Orlov about the structure of equivalences is no longer true in the non-projective setting.

One of the main topics treated in Section 4.1 is the definition of Chern classes for coherent sheaves on compact complex varieties. These classes will be used in the forthcoming sections.

An extensive introduction to derived categories and derived functors with particular attention to the non-algebraic case is included for the convenience of the reader. In Section 4.5 we study compact complex manifolds derived equivalent to a complex  $K3$  surface.

##### 4.1. Coherent sheaves, Chern classes and locally free resolutions

In the algebraic setting the abelian category  $\mathbf{Coh}(X)$  of a smooth projective variety is a very strong invariant. Indeed the following classical result of Gabriel shows that it actually encodes all the geometric information about the variety.

**Theorem 4.1** ([28]). *Let  $X$  and  $Y$  be smooth projective varieties such that  $\mathbf{Coh}(X) \cong \mathbf{Coh}(Y)$  as abelian categories. Then there exists an isomorphism  $X \cong Y$ .*

This is not the case for generic analytic  $K3$  surfaces. Indeed a result of Verbitsky shows that coherent sheaves are not very interesting for varieties of this type. The original proof in [87] is quite involved and we plan to simplify it in a future paper.

**Theorem 4.2** ([87], Theorem 1.1). *Let  $X$  and  $Y$  be  $K3$  surfaces such that  $\rho(X) = \rho(Y) = 0$ . Then  $\mathbf{Coh}(X) \cong \mathbf{Coh}(Y)$  as abelian categories.*

The same result was proved in [88] for generic (non-projective) complex tori following very closely the approach in [87].

Coherent sheaves on non-algebraic  $K3$  surfaces share some useful properties with coherent sheaves on algebraic  $K3$  surfaces. More precisely, due to [80] any coherent sheaf on a projective algebraic variety admits a finite locally free resolution. In [77] Schuster proved that this is true for any compact complex surface. In the special case of  $K3$  surfaces, this reads as follows:

**Theorem 4.3.** *Any coherent sheaf  $\mathcal{E} \in \mathbf{Coh}(X)$  on a  $K3$  surface  $X$  admits a locally free resolution of finite length.*

Note that this does not hold for any analytic variety as it was shown in [89]. As an easy consequence of Schuster’s theorem, it makes perfect sense to define the Chern character of any coherent sheaf  $\mathcal{E} \in \mathbf{Coh}(X)$  for a  $K3$  surface  $X$ . Indeed, one considers a locally free resolution of finite length of  $\mathcal{E}$  and takes the (well-defined) alternating sum of the Chern characters of the locally free sheaves appearing in the resolution. Unfortunately this is not sufficient for the purposes of the forthcoming sections since we will also need to calculate the Chern character of coherent sheaves (or complexes) on the product of two  $K3$  surfaces, where Theorem 4.3 is not known to hold true. For this we need to define Chern classes in a different way.

Let  $X$  be, more generally, a compact complex manifold. Chern classes for coherent sheaves on  $X$  with values in the Hodge cohomology of  $X$  (i.e.  $\bigoplus_p H^p(X, \Omega_X^p)$ ) were defined by O’Brian, Toledo, and Tong in [65], substituting locally free resolutions by what they called *twisted cochain complexes*. They also proved Hirzebruch–Riemann–Roch and Grothendieck–Riemann–Roch formulas for these classes [66–68]. Notice that this definition is better suited to the setting of derived categories, due to the close relationship with Hochschild homology (see e.g. [20,21], as pointed out in [8]).

In our work we will follow a different definition by Green contained in his unpublished Ph.D. thesis [35] (for a review of it, see [86]). In Green’s approach these classes are defined via *complexes of simplicial vector bundles* and they take values in the de Rham cohomology of the manifold. Moreover, due to [86, Thm. 2], they coincide with Chern classes defined by Atiyah–Hirzebruch by considering locally free resolutions in the real-analytic setting.

We present a few details of the construction but the reader is also strongly encouraged to consult the original works for precise definitions and results. In particular, let  $\mathcal{E}$  be a coherent sheaf of  $\mathcal{O}_X$ -modules and consider the sheaf  $\mathcal{O}_X^{\text{an}}$  of germs of complex-valued real-analytic functions on  $X$ . Denote by  $\mathcal{E}^{\text{an}}$  the  $\mathcal{O}_X^{\text{an}}$ -module  $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\text{an}}$ . Then  $\mathcal{E}^{\text{an}}$  is a coherent  $\mathcal{O}_X^{\text{an}}$ -module.

**Proposition 4.4** ([3], Proposition 2.6). *Let  $S$  be a coherent sheaf of  $\mathcal{O}_X^{\text{an}}$ -modules. Then there exists a resolution of  $S$  over  $\mathcal{O}_X^{\text{an}}$*

$$0 \rightarrow L_k \rightarrow \dots \rightarrow L_0 \rightarrow S \rightarrow 0,$$

where  $L_i$  is a locally free  $\mathcal{O}_X^{\text{an}}$ -module, for all  $i$ .

According to [3], the previous proposition allows one to define Chern classes of coherent sheaves on  $X$  with values in  $H^*(X, \mathbb{Q})$ . All the usual properties of Chern classes hold with this definition. For example: commutativity with flat pull-backs, additivity under exact sequences of sheaves (which in particular implies that Chern classes are actually well-defined at the level of Grothendieck group  $K(X) := K(\mathbf{Coh}(X))$  of the abelian category  $\mathbf{Coh}(X)$ ) and Whitney’s formula. Moreover, Riemann–Roch formulas (Theorem 4.20) for Green’s Chern classes and a comparison between the various definitions are contained in [35,86]. Clearly in the case of either compact complex surfaces or projective complex manifolds all these definitions coincide with the classical one via locally free resolutions in the complex analytic category.

**Definition 4.5.** For  $e \in K(X)$ , the *Mukai vector* of  $e$  is the vector  $v(e) := \text{ch}(e) \cdot \sqrt{\text{td}(X)} \in H^*(X, \mathbb{Q})$ .

Just to consider the easiest example, suppose  $E$  to be a locally free sheaf on a  $K3$  surface  $X$  with rank  $r := \text{rk } E$ , first Chern class  $c_1 := c_1(E)$  and second Chern class  $c_2 := c_2(E)$ . If  $e$  is the class of  $E$  in  $K(X)$ , explicit calculations yields

$$v(e) = (r, c_1, c_1^2/2 - c_2 + r).$$

From this we deduce that if  $X$  is a generic analytic  $K3$  surface (i.e.  $\text{NS}(X) = 0$ ), the Mukai vector of any  $e \in K(X)$  is  $v(e) = (r, 0, s)$ . For the rest of this paper, given a smooth compact complex variety  $X$  and  $\mathcal{E} \in \mathbf{Coh}(X)$ , we denote by  $[\mathcal{E}]$  the class of  $\mathcal{E}$  in the Grothendieck group of  $X$ .

**Remark 4.6.** Since  $\mathbf{Coh}(X)$  does not univocally encode the geometry of a compact complex manifold  $X$ , one should work with larger categories of sheaves. Unfortunately quasi-coherent sheaves are not the best choice: many different definitions of quasi-coherent sheaves are present in the literature and none of them seems to be the definitive one. Moreover the relations between these definitions have not been completely explored at the moment (see the discussion at the end of Section 5 of [6] for more details). In Section 4.3 we will work with the abelian category of  $\mathcal{O}_X$ -modules.

#### 4.2. Derived categories and functors

For the convenience of the reader we briefly recall the algebraic construction of the derived category of an abelian category. Since we tend to be very brief in this section, the reader interested in more details and in all the proofs is encouraged to consult [34,43]. On the other hand the expert reader can easily skip this rather formal section.

Let  $\mathbf{A}$  be an abelian category. Our main examples will be the abelian category  $\mathcal{O}_X\text{-Mod}$  of  $\mathcal{O}_X$ -modules and its full abelian subcategory  $\mathbf{Coh}(X)$  of coherent sheaves, for a complex manifold  $X$ . A complex  $M^\bullet$  over  $\mathbf{A}$  is a pair  $(\{M^i\}_{i \in \mathbb{Z}}, \{d^i\}_{i \in \mathbb{Z}})$  such that  $M^i \in \mathbf{A}$  and  $d^i : M^i \rightarrow M^{i+1}$  is a morphism such that  $d^j \circ d^{j-1} = 0$ , for any  $j \in \mathbb{Z}$ . For simplicity, a complex will be sometimes denoted by  $M^\bullet$  or by the pair  $(M^\bullet, d^\bullet)$ . It can be also useful to think of it as a diagram

$$\dots \rightarrow M^{p-1} \xrightarrow{d^{p-1}} M^p \xrightarrow{d^p} M^{p+1} \rightarrow \dots,$$



where  $p \in \mathbb{Z}$ . A morphism of complexes is a set of arrows  $\{f^i\}_{i \in \mathbb{Z}}$  as in the following diagram

$$\begin{array}{ccccccc} \dots & \xrightarrow{d_{M^\bullet}^{i-2}} & M^{i-1} & \xrightarrow{d_{M^\bullet}^{i-1}} & M^i & \xrightarrow{d_{M^\bullet}^i} & M^{i+1} \xrightarrow{d_{M^\bullet}^{i+1}} \dots \\ & & \downarrow f^{i-1} & & \downarrow f^i & & \downarrow f^{i+1} \\ \dots & \xrightarrow{d_{L^\bullet}^{i-2}} & L^{i-1} & \xrightarrow{d_{L^\bullet}^{i-1}} & L^i & \xrightarrow{d_{L^\bullet}^i} & L^{i+1} \xrightarrow{d_{L^\bullet}^{i+1}} \dots \end{array}$$

and such that, for any  $i \in \mathbb{Z}$ ,  $f^i \circ d_{M^\bullet}^{i-1} = d_{L^\bullet}^{i-1} \circ f^{i-1}$ . In this way we obtain the abelian category of complexes  $C(\mathbf{A})$ .

A morphism of complexes is a *quasi-isomorphism* if, for any  $p \in \mathbb{Z}$ , it induces isomorphisms between all the cohomology objects

$$H^p(M^\bullet) := \ker d_{M^\bullet}^p / \text{im } d_{M^\bullet}^{p-1}$$

of the complex  $M^\bullet$ . We define the derived category  $D(\mathbf{A})$  as the localization (see [29]) of the category of complexes with respect to the class of quasi-isomorphisms. So we get a category with the same objects as  $C(\mathbf{A})$  and in which we formally invert all quasi-isomorphism.

This quick definition is not very explicit. To get a description of the morphisms in  $D(\mathbf{A})$  we should first consider another category  $\text{Kom}(\mathbf{A})$  whose construction goes as follows. First observe that a morphism of complexes  $f : M^\bullet \rightarrow L^\bullet$  is *null-homotopic* if there are morphisms  $r^p : M^p \rightarrow L^{p-1}$  such that

$$f^p = d_{L^\bullet}^{p-1} \circ r^p + r^{p+1} \circ d_{M^\bullet}^p,$$

for all  $p \in \mathbb{Z}$ . The objects of  $\text{Kom}(\mathbf{A})$  are the complexes in  $C(\mathbf{A})$  while the morphisms are obtained from the ones in  $C(\mathbf{A})$  by taking the quotient modulo null-homotopic morphisms. In  $\text{Kom}(\mathbf{A})$  the quasi-isomorphisms form a *localizing class* (in the sense of Verdier) and then it is possible to have an explicit description of the morphisms in  $D(\mathbf{A})$  as “fractions”, essentially in the same way as for the product in a localized ring.

The basic properties of derived categories are summarized in the following proposition.

**Proposition 4.7.** (i) *The category  $D(\mathbf{A})$  is additive.*

(ii) *The canonical functor  $Q : C(\mathbf{A}) \rightarrow D(\mathbf{A})$  is additive and makes all quasi-isomorphisms invertible and is universal among all functors  $C(\mathbf{A}) \rightarrow \mathbf{C}$  having this property, where  $\mathbf{C}$  is any category.*

(iii) *The canonical functor  $\mathbf{A} \rightarrow D(\mathbf{A})$ , which maps an object  $A \in \mathbf{A}$  to the complex with all zero besides  $A$  in degree 0, is fully faithful.*

For the geometric applications we have in mind, we will be interested in some full subcategories of  $D(\mathbf{A})$ .

**Example 4.8.** (i) We denote by  $D^+(\mathbf{A})$  (respectively  $D^-(\mathbf{A})$  and  $D^b(\mathbf{A})$ ), the full additive subcategory of complexes  $M^\bullet$  such that  $M^n = 0$ , for all  $n \gg 0$  (respectively  $n \ll 0$  and  $|n| \gg 0$ ). Such a category is equivalent to the category of complexes where  $H^n(M^\bullet) = 0$ , for all  $n \gg 0$  (respectively  $n \ll 0$  and  $|n| \gg 0$ ).

(ii)  $D_{\mathbf{A}'}^*(\mathbf{A})$  is the full additive subcategory of complexes  $M^\bullet$  such that  $H^n(M^\bullet) \in \mathbf{A}'$ , for all  $n \in \mathbb{Z}$ , where  $\mathbf{A}'$  is a full *thick* subcategory of  $\mathbf{A}$  (i.e. any extension in  $\mathbf{A}$  of two objects of  $\mathbf{A}'$  is in  $\mathbf{A}'$ ) and  $*$  =  $\emptyset, +, -, b$ . For example  $\text{Coh}(X) \subseteq \mathcal{O}_X\text{-Mod}$  is a thick subcategory.

Denote by  $[1] : D(\mathbf{A}) \rightarrow D(\mathbf{A})$  the *shift functor* which maps a complex  $(M^\bullet, d_{M^\bullet}^\bullet)$  to the complex  $(M^\bullet[1], d_{M^\bullet[1]}^\bullet)$ , defined by  $(M^\bullet[1])^i = M^{i+1}$  and  $d_{M^\bullet[1]}^i = -d_{M^\bullet}^{i+1}$ . Notice that a morphism of complexes is mapped to the same morphisms with degrees shifted by 1. A *standard triangle* in  $D(\mathbf{A})$  is a sequence

$$X^\bullet \xrightarrow{Q(p)} Y^\bullet \xrightarrow{Q(q)} Z^\bullet \xrightarrow{\partial \varepsilon} X[1]^\bullet,$$

where  $Q : C(\mathbf{A}) \rightarrow D(\mathbf{A})$  is the canonical functor,

$$\varepsilon : 0 \rightarrow X^\bullet \xrightarrow{p} Y^\bullet \xrightarrow{q} Z^\bullet \rightarrow 0$$

is a short exact sequence of complexes in  $C(\mathbf{A})$ , and  $\partial \varepsilon$  is a special morphism in  $D(\mathbf{A})$  functorial in  $\varepsilon$  and which lifts the connecting morphism  $H^i(Z^\bullet) \rightarrow H^{i+1}(X^\bullet)$  in the long exact cohomology sequence associated with  $\varepsilon$ . A *triangle*

in  $D(\mathbf{A})$  is a sequence

$$X^\bullet \rightarrow Y^\bullet \rightarrow Z^\bullet \rightarrow X^\bullet[1]$$

isomorphic to a standard triangle.

More generally, a *triangulated category* is an additive category  $\mathbf{T}$  endowed with an autoequivalence  $X \mapsto X[1]$  and a class of sequences (called *triangles*) of the form

$$X \rightarrow Y \rightarrow Z \rightarrow X[1]$$

briefly denoted by  $(X, Y, Z)$  which are stable under isomorphisms and satisfy a list of axioms (see, for example, [62, Ch. 1] for a nice presentation). If  $\mathbf{T}$  is a triangulated category, a *full triangulated subcategory* of  $\mathbf{T}$  is a full subcategory  $\mathbf{U} \subseteq \mathbf{T}$  such that  $\mathbf{U}[1] = \mathbf{U}$  and, whenever we have a triangle  $(X, Y, Z)$  of  $\mathbf{T}$  such that  $X$  and  $Z$  belong to  $\mathbf{U}$ , then there is  $Y' \in \mathbf{U}$  isomorphic to  $Y$ . A full triangulated subcategory is again triangulated in the natural way. For example, all the subcategories of  $D(\mathbf{A})$  considered in Example 4.8 are full triangulated subcategories.

To a triangulated category  $\mathbf{T}$  is naturally associated an abelian group called *Grothendieck group* and denoted by  $K(\mathbf{T})$ . It is defined as the quotient of the free abelian group generated by the isomorphism classes  $[X]$  of objects of  $\mathbf{T}$  divided by the subgroup generated by the relations

$$[X] - [Y] + [Z],$$

where  $(X, Y, Z)$  runs through the triangles of  $\mathbf{T}$ .

We have just seen how to pass from an abelian category to a triangulated one. Later we will need to invert this process. This is related to the notion of *t-structure*. Indeed, a *t-structure* (see [5]) on a triangulated category  $\mathbf{T}$  is a strictly (i.e. closed under isomorphisms) full additive subcategory  $\mathbf{F} \subseteq \mathbf{T}$  such that  $\mathbf{F}[1] \subseteq \mathbf{F}$  and, if

$$\mathbf{F}^\perp := \{G \in \mathbf{T} : \text{Hom}_{\mathbf{T}}(F, G) = 0, \text{ for all } F \in \mathbf{F}\},$$

then for any non-zero  $E \in \mathbf{T}$ , there exists a triangle

$$F \rightarrow E \rightarrow G \rightarrow F[1]$$

with  $F \in \mathbf{F}$  and  $G \in \mathbf{F}^\perp$ . The *heart*  $\mathbf{A} := \mathbf{F} \cap \mathbf{F}^\perp[1]$  of the *t-structure* is a full abelian subcategory of  $\mathbf{T}$ , with the short exact sequences being precisely the triangles in  $\mathbf{T}$  whose vertices are objects of  $\mathbf{A}$ . A *t-structure* is called *bounded* if

$$\mathbf{T} = \bigcup_{i,j} (\mathbf{F}[i] \cap \mathbf{F}^\perp[j]).$$

Note that for a bounded *t-structure* one has  $K(\mathbf{T}) \cong K(\mathbf{A})$ .

Let us now briefly recall how one can produce functors between derived categories starting from functors between the corresponding abelian categories. We will follow Deligne’s approach as outlined in [49]. Let  $\mathbf{A}$  and  $\mathbf{B}$  be abelian categories and let  $F : \mathbf{A} \rightarrow \mathbf{B}$  be an additive functor. Take  $M^\bullet \in D(\mathbf{A})$  and define a functor

$$(\mathbf{r}F)(-, M^\bullet) : (D(\mathbf{B}))^{\text{op}} \rightarrow \mathbf{Mod}\text{-}\mathbb{Z}$$

in the following way: for any  $L^\bullet \in D(\mathbf{B})$ .

$$(\mathbf{r}F)(L^\bullet, M^\bullet) = \left\{ (f, s) : \begin{array}{l} (1) f \in \text{Hom}_{D(\mathbf{B})}(L^\bullet, F(N^\bullet)) \\ (2) s \in \text{Hom}_{\text{Kom}(\mathbf{A})}(M^\bullet, N^\bullet) \text{ is a quasi-isomorphism} \end{array} \right\} / \sim,$$

where  $(f, s) \sim (g, t)$  if and only if there exists  $(h, u)$  making all obvious diagrams commutative. If the functor  $(\mathbf{r}F)(-, M^\bullet)$  is representable, then we denote by  $\mathbf{R}F(M^\bullet)$  the object representing it and we say that  $\mathbf{R}F$  is defined at  $M^\bullet \in D(\mathbf{A})$ . We call the functor  $\mathbf{R}F$  constructed in this way (and defined in general only on a subcategory of  $D(\mathbf{A})$ ) the *right derived functor* of  $F$ . Of course, the above construction can be applied without changes to arbitrary functors  $F : \text{Kom}(\mathbf{A}) \rightarrow \text{Kom}(\mathbf{B})$ . Moreover, one can dually define *left derived functors*  $\mathbf{L}F : D(\mathbf{A}) \rightarrow D(\mathbf{B})$  by inverting the direction of the arrows in the above definition. The link between the previous definition and the classical one is illustrated by the following result.

**Proposition 4.9.** *Suppose that  $\mathbf{A}$  has enough injectives and  $M^\bullet$  is left bounded. Then  $\mathbf{R}F$  is defined at  $M^\bullet$  and we have*

$$\mathbf{R}F(M^\bullet) = F(I^\bullet),$$

where  $M^\bullet \rightarrow I^\bullet$  is a quasi-isomorphism with a left bounded complex with injective components.

**Example 4.10.** If  $M^\bullet = M \in \mathbf{A}$  is concentrated in degree zero, then  $I^\bullet$  may be chosen to be an injective resolution of  $M$  and we find that

$$H^n(\mathbf{R}F(M)) = (\mathbf{R}^n F)(M),$$

the  $n$ th right derived functor of  $F$  in the classical sense of Cartan–Eilenberg (see [38]). For example, if  $F = \text{Hom}(N, -)$ , with  $N \in \mathbf{A}$ , then, for  $M \in \mathbf{A}$ , we have the chain of natural isomorphisms

$$\text{Ext}_{\mathbf{A}}^n(N, M) \cong H^n(\mathbf{R} \text{Hom}(N, M)) \cong \text{Hom}_{D(\mathbf{A})}(N, M[n]), \tag{4.1}$$

for all  $n \in \mathbb{Z}$ .

An additive functor  $F : \mathbf{T} \rightarrow \mathbf{T}'$  between two triangulated categories is called *exact* if there exists a functorial isomorphism

$$\phi_X : F(X[1]) \xrightarrow{\sim} (F(X))[1]$$

such that the sequence

$$F(X) \xrightarrow{F(w)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{(\phi_X)F(w)} (F(X))[1]$$

is a triangle of  $\mathbf{T}'$ , for any triangle of  $\mathbf{T}$ .

For the rest of this paper, we will write  $M$  and  $f$  instead of  $M^\bullet$  and  $f^\bullet$  to denote complexes and morphisms of complexes. Moreover all functors will be supposed to be exact.

### 4.3. Derived categories of complex manifolds

In this section we study the derived and triangulated categories associated with the abelian categories of coherent sheaves on complex manifolds. In particular,  $X$  will always be a compact complex manifold.

**Definition 4.11.** The (*bounded*) *derived category* of  $X$ , denoted by  $D^b(X)$  is defined as the triangulated subcategory  $D_{\mathbf{Coh}}^b(\mathcal{O}_X\text{-Mod})$  of  $D(\mathcal{O}_X\text{-Mod})$  whose objects are bounded complexes of  $\mathcal{O}_X$ -modules with cohomologies in  $\mathbf{Coh}(X)$ .

**Example 4.12.** For any closed point  $x \in X$ , the skyscraper sheaf  $\mathcal{O}_x$  is in  $D^b(X)$ . Moreover,

$$\text{Hom}(\mathcal{O}_x, \mathcal{O}_x[i]) \cong \begin{cases} \mathbb{C} & \text{if } i \in \{0, 2\} \\ \mathbb{C} \oplus \mathbb{C} & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Any object  $\mathcal{E} \in D^b(X)$  such that  $\text{Hom}(\mathcal{E}, \mathcal{E}[i])$  is as in (4.2) is called *semi-rigid*.

Now recall that given an abelian category  $\mathbf{A}$ ,  $A \in \text{Ob}(\mathbf{A})$  is *minimal* if any non-trivial surjective morphism  $A \rightarrow B$  in  $\mathbf{A}$  is an isomorphism. It is an easy exercise to prove that the minimal objects of  $\mathbf{Coh}(X)$  are the skyscraper sheaves.

**Example 4.13.** In [79] a special class of objects was introduced. An object  $\mathcal{E}^\bullet \in D^b(X)$  is a *spherical object* if  $\mathcal{E}^\bullet \otimes \omega_X \cong \mathcal{E}^\bullet$ , where  $\omega_X$  is the dualizing sheaf of  $X$  and

$$\text{Hom}(\mathcal{E}^\bullet, \mathcal{E}^\bullet[i]) \cong \begin{cases} \mathbb{C} & \text{if } i \in \{0, \dim X\} \\ 0 & \text{otherwise.} \end{cases}$$

In particular if  $X$  is a K3 surface, any line bundle (and hence also  $\mathcal{O}_X$ ) is a spherical object of  $D^b(X)$ .

The definition of  $D^b(X)$  may seem not appropriate. A better candidate to work with would be the derived category  $D^b(\mathbf{Coh}(X))$ . As we will see in the next section, unfortunately this category is not the natural place to derive functors and to verify Serre duality. Clearly there is a natural exact functor

$$F : D^b(\mathbf{Coh}(X)) \rightarrow D^b(X). \tag{4.3}$$

In particular  $\mathbf{Coh}(X)$  is the heart of a bounded  $t$ -structure on  $D^b(X)$  and there is a natural identification  $K(X) \cong K(D^b(X))$  (note that, via this identification, the Chern classes and the Chern character are well-defined for objects in  $D^b(X)$ ). The best situation is when this functor is actually an equivalence. In the algebraic case this is a well-known result (see, for example, Corollary 2.2.2.1 of Exposé II in [47]).

**Proposition 4.14.** *Let  $X$  be a smooth projective complex manifold. Then the functor  $F$  in (4.3) is an equivalence.*

In the purely analytic case the situation is more complicated due to the absence of a good notion of quasi-coherent sheaves (see Remark 4.6). But in the surface case the result is still true via Serre duality (see Section 4.5).

**Proposition 4.15** ([9], Proposition 5.2.1). *Let  $X$  be a smooth compact complex surface. Then the functor  $F$  in (4.3) is an equivalence.*

**Proof.** We denote by  $\text{Ext}_1^n$  and  $\text{Ext}_2^n$  respectively the Ext-groups  $\text{Ext}_{\mathbf{Coh}(X)}^n$  and  $\text{Ext}_{\mathcal{O}_X\text{-Mod}}^n$  in the categories  $\mathbf{Coh}(X)$  and  $\mathcal{O}_X\text{-Mod}$ , that is, by (4.1), the groups  $\text{Hom}(-, -[n])$  in the corresponding derived categories.

The result follows using [47, Lemma 2.1.3] once we show that for any  $\mathcal{E}, \mathcal{F} \in \mathbf{Coh}(X)$ , the natural morphism  $f_n : \text{Ext}_1^n(\mathcal{E}, \mathcal{F}) \rightarrow \text{Ext}_2^n(F(\mathcal{E}), F(\mathcal{F}))$  is an isomorphism for any  $n$ . For  $n = 0$  this is true by the very definition of the functor  $\text{Hom}$  in both categories. For  $n = 1$  this property is trivially verified since the extension of coherent sheaves is still coherent.

To show that  $f_n$  is invertible for  $n \neq 1$ , we apply once more [47, Lemma 2.1.3] which asserts that this happens if for any  $\mathcal{E}, \mathcal{F} \in \mathcal{O}_X\text{-Mod}$  and any  $f \in \text{Ext}_2^n(\mathcal{E}, \mathcal{F})$ , there exists a surjective morphism  $\mathcal{E}' \rightarrow \mathcal{E}$  such that  $f$  is sent to zero by the natural morphism  $\text{Ext}_2^n(\mathcal{E}, \mathcal{F}) \rightarrow \text{Ext}_2^n(\mathcal{E}', \mathcal{F})$ .

If  $n > 2$ , Serre duality discussed in Section 4.5 for analytic varieties implies that  $\text{Ext}_2^n(\mathcal{E}, \mathcal{F}) = 0$  and so there is nothing to prove. Hence we just need to consider the case  $n = 2$ .

Let us first show that for any  $\mathcal{E}, \mathcal{F} \in \mathbf{Coh}(X)$  and any  $x \in X$ , there is  $n$  such that  $\text{Ext}_2^n(m_x^n \mathcal{E}, \mathcal{F}) = 0$ , where  $m_x$  is the maximal ideal of  $x$  in  $\mathcal{O}_X$ . Due to Serre duality, we can just show that for  $n \gg 0$  one has  $\text{Hom}(\mathcal{G}, m_x^n \mathcal{E}) = 0$ , where  $\mathcal{G} = \mathcal{F} \otimes \omega_X$  and  $\omega_X$  is the dualizing sheaf of  $X$ .

Since  $\text{Hom}(\mathcal{G}, m_x^n \mathcal{E})$  is finite-dimensional over  $\mathbb{C}$ , it is sufficient to show that for  $a \in \mathbb{N}$  there is  $b > a$  such that  $\text{Hom}(\mathcal{G}, m_x^a \mathcal{E}) \neq \text{Hom}(\mathcal{G}, m_x^b \mathcal{E})$ . If we pick  $f : \mathcal{G} \rightarrow m_x^a \mathcal{E}$  then it is an easy check using stalks that there is  $b$  such that  $\text{im}(f_x) \not\subset m_x^b \mathcal{E}_x$ . Hence  $f \notin \text{Hom}(\mathcal{G}, m_x^b \mathcal{E})$ .

Now we take  $x, y \in X$  such that  $x \neq y$  and we choose  $n \in \mathbb{N}$  such that  $\text{Ext}_2^n(m_x^n \mathcal{E}, \mathcal{F}) = \text{Ext}_2^n(m_y^n \mathcal{E}, \mathcal{F}) = 0$ . The canonical map  $m_x^n \mathcal{E} \oplus m_y^n \mathcal{E} \rightarrow \mathcal{F}$  is clearly surjective. This concludes the proof.  $\square$

Going back to the case of generic analytic K3 surfaces  $X$ , we can now state the following crucial property of  $D^b(X)$  which completely classifies its spherical objects. This will be important when we will study stability conditions on the derived category of these K3 surfaces.

**Lemma 4.16** ([45], Lemma 3.1). *Let  $X$  be a generic analytic K3 surface. The trivial line bundle  $\mathcal{O}_X$  is up to shift the only spherical object in  $D^b(X)$ .*

#### 4.4. Derived functors on complex manifolds

Let  $X$  and  $Y$  be compact complex manifolds. Consider their corresponding derived categories. As we pointed out in Section 4.2, not all functors can be derived. Nevertheless, the following result shows that all functors with a geometric meaning are subject to this procedure.

**Proposition 4.17.** *Let  $f, g : X \rightarrow Y$  be morphisms and assume  $f$  to be proper. Then the following functors are defined:*

$$\begin{aligned} \mathbf{R}\mathcal{H}om^\bullet &: D^b(X)^{op} \times D^b(X) \longrightarrow D^b(X), \\ - \overset{\mathbf{L}}{\otimes} - &: D^b(X) \times D^b(X) \longrightarrow D^b(X), \\ \mathbf{L}g^* &: D^b(Y) \longrightarrow D^b(X), \\ \mathbf{R}f_* &: D^b(X) \longrightarrow D^b(Y), \\ \mathbf{R}\mathcal{H}om^\bullet &: D^b(X)^{op} \times D^b(X) \longrightarrow D^b(\mathbf{vect}), \end{aligned}$$

where we denote by **vect** the abelian category of finite-dimensional  $\mathbb{C}$ -vector spaces.

**Proof.** According to [82], the functors listed in the proposition are all well-defined if we consider the corresponding unbounded derived categories of modules  $D(\mathcal{O}_X\text{-Mod})$ ,  $D(\mathcal{O}_Y\text{-Mod})$  and  $D(\mathbf{Vect})$  (where we denote by **Vect** the abelian category of  $\mathbb{C}$ -vector spaces). To prove the proposition we only have to show that, by restricting the domain to the corresponding derived categories of bounded complexes with coherent cohomology, the images of these functors are still in the bounded derived categories of complexes with coherent cohomology. But these are well-known results (see, for example, [36, Ch. 5]).  $\square$

To give to the reader a first impression of the relationships between the derived functors just defined, we list here a few of their properties whose proofs are, for example, in [82]. As before,  $X, Y$  and  $Z$  are compact complex manifolds while  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are morphisms. Then there exist the following natural isomorphisms of functors:

- Assuming  $f$  and  $g$  proper,  $\mathbf{R}(g_* \circ f_*) \cong \mathbf{R}g_* \circ \mathbf{R}f_*$ .
- $\mathbf{L}(f^* \circ g^*) \cong \mathbf{L}f^* \circ \mathbf{L}g^*$ .
- $\mathbf{R}\mathcal{H}om^\bullet(\mathcal{F}, \mathcal{G}) \cong \mathbf{R}\Gamma(X, \mathbf{R}\mathcal{H}om(\mathcal{F}, \mathcal{G}))$  as bi-functors.
- $\mathbf{R}f_*\mathcal{H}om^\bullet(\mathcal{F}, \mathcal{G}) \cong \mathbf{R}\mathcal{H}om(\mathbf{R}f_*(\mathcal{F}), \mathbf{R}f_*(\mathcal{G}))$ .
- Projection Formula: assuming  $f$  proper,  $\mathbf{R}f_*(\mathcal{F}) \overset{bfL}{\otimes} \mathcal{G} \cong \mathbf{R}f_*(\mathcal{F} \overset{bfL}{\otimes} \mathbf{L}f^*(\mathcal{G}))$ .
- Assuming  $f$  proper,  $\mathcal{H}om^\bullet(\mathbf{L}f^*(\mathcal{F}), \mathcal{G}) \cong \mathcal{H}om(\mathcal{F}, \mathbf{R}f_*(\mathcal{G}))$ . In other words,  $\mathbf{L}f^*$  is left adjoint to  $\mathbf{R}f_*$ .
- $\mathbf{L}f^*(\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{G}) \cong \mathbf{L}f^*(\mathcal{F}) \overset{\mathbf{L}}{\otimes} \mathbf{L}f^*(\mathcal{G})$ .
- $\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{G} \cong \mathcal{G} \overset{\mathbf{L}}{\otimes} \mathcal{F}$  and  $\mathcal{F} \overset{\mathbf{L}}{\otimes} (\mathcal{G} \overset{\mathbf{L}}{\otimes} \mathcal{H}) \cong (\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{G}) \overset{\mathbf{L}}{\otimes} \mathcal{H}$ .
- $\mathbf{R}\mathcal{H}om^\bullet(\mathcal{F}, \mathcal{G}) \overset{\mathbf{L}}{\otimes} \mathcal{H} \cong \mathbf{R}\mathcal{H}om^\bullet(\mathcal{F}, \mathcal{G} \overset{\mathbf{L}}{\otimes} \mathcal{H})$ .
- $\mathbf{R}\mathcal{H}om^\bullet(\mathcal{F}, \mathbf{R}\mathcal{H}om^\bullet(\mathcal{G}, \mathcal{H})) \cong \mathbf{R}\mathcal{H}om^\bullet(\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{G}, \mathcal{H})$ .

Moreover, if  $u : Y' \rightarrow Y$  is a flat morphism fitting in the following commutative diagram

$$\begin{array}{ccc} X \times_Y Y' & \xrightarrow{v} & X \\ g \downarrow & & \downarrow f \\ Y' & \xrightarrow{u} & Y, \end{array}$$

then there exists a functorial isomorphism

$$u^*\mathbf{R}f_*(\mathcal{F}) \cong \mathbf{R}g_*v^*(\mathcal{F}).$$

Such a property is usually called Flat Base Change Theorem.

**Definition 4.18.** (i) Let  $X$  and  $Y$  be compact complex manifolds. A functor  $F : D^b(X) \rightarrow D^b(Y)$  is of *Fourier–Mukai type* (or a *Fourier–Mukai functor*) if there exists  $\mathcal{E} \in D^b(X \times Y)$  and an isomorphism of functors  $F \cong \Phi_{\mathcal{E}}$ , where, denoting by  $p : X \times Y \rightarrow Y$  and  $q : X \times Y \rightarrow X$  the natural projections,  $\Phi_{\mathcal{E}} : D^b(X) \rightarrow D^b(Y)$  is the exact functor defined by

$$\Phi_{\mathcal{E}} := \mathbf{R}p_*(\mathcal{E} \overset{\mathbf{L}}{\otimes} q^*(-)). \tag{4.4}$$

The complex  $\mathcal{E}$  is a kernel of  $F$ .

(ii) Two compact complex manifolds  $X$  and  $Y$  are *Fourier–Mukai partners* if there exists a Fourier–Mukai transform  $\Phi : D^b(X) \rightarrow D^b(Y)$ .

It is a nice exercise to prove that the equivalences in the following example are of Fourier–Mukai type.

**Example 4.19.** Here  $X$  and  $Y$  are always meant to be compact complex smooth surfaces.

- (i) Let  $f : X \xrightarrow{\sim} Y$  be an isomorphism. Then  $f^* : D^b(Y) \xrightarrow{\sim} D^b(X)$  is an equivalence.
- (ii) Let  $\mathcal{E}$  be a spherical object (see Example 4.13). Consider the *spherical twist*  $T_{\mathcal{E}} : D^b(X) \rightarrow D^b(X)$  that sends  $\mathcal{F} \in D^b(X)$  to the cone of

$$\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \rightarrow \mathcal{F}.$$

The kernel of  $T_{\mathcal{E}}$  is given by the cone of the natural map  $\mathcal{E}^\vee \boxtimes \mathcal{E} \rightarrow \mathcal{O}_\Delta$ , where  $\mathcal{O}_\Delta := \Delta_* \mathcal{O}_X$  and  $\Delta$  is the diagonal embedding  $X \hookrightarrow X \times X$  (notice that here we need the existence of locally free resolutions as stated in Theorem 4.3).

(iii) For any line bundle  $L \in \mathrm{Pic}(X)$  we have the *line bundle twist*  $(-) \otimes L : D^b(X) \xrightarrow{\sim} D^b(X)$ .

**Theorem 4.20** ([68,86]). *Let  $X$  and  $Y$  be compact complex manifolds and let  $f : X \rightarrow Y$  be a holomorphic map. Then, for all  $\mathcal{E} \in D^b(X)$ , Grothendieck–Riemann–Roch formula holds in  $H^*(Y, \mathbb{Q})$ :*

$$\mathrm{ch}(\mathbf{R}f_* \mathcal{E}) \cdot \mathrm{td}(Y) = f_*(\mathrm{ch}(\mathcal{E}) \cdot \mathrm{td}(X)).$$

In particular, if  $Y$  is a point, for all  $\mathcal{E} \in D^b(X)$ , Hirzebruch–Riemann–Roch formula holds:

$$\chi(\mathcal{E}) := \sum_i (-1)^i \dim H^i(X, \mathcal{E}) = (\mathrm{ch}(\mathcal{E}) \cdot \mathrm{td}(X))_{\dim X},$$

where  $(-)_{\dim X}$  denotes the  $\dim X$ -component of a class in  $H^*(X, \mathbb{Q})$ .

As in the projective case, the Chern class of a coherent sheaf on a  $K3$  surface is integral. The previous theorem can be used to prove that  $\mathrm{ch}(\mathcal{E})$  (and then  $v([\mathcal{E}])$ ) is integral for any  $\mathcal{E} \in D^b(X \times X)$ , where  $X$  is a  $K3$  surface, following line-by-line the original proof of Mukai in [43, Lemma 10.6] of the same result in the projective case. Moreover the Hirzebruch–Riemann–Roch formula can be rewritten for  $K3$  surfaces as

$$\chi(\mathcal{E}, \mathcal{F}) := \sum_i (-1)^i \dim \mathrm{Hom}_{D^b(X)}(\mathcal{E}, \mathcal{F}[i]) = -\langle v([\mathcal{E}]), v([\mathcal{F}]) \rangle, \tag{4.5}$$

for any  $\mathcal{E}, \mathcal{F} \in D^b(X)$  (here  $\langle -, - \rangle$  is the Mukai pairing introduced in (2.2)). The pairing  $\chi(-, -)$  will be sometimes called *Euler pairing*.

Given two compact complex manifolds  $X$  and  $Y$ , let  $\mathrm{Eq}(D^b(X), D^b(Y))$  be the set of equivalences between the derived categories  $D^b(X)$  and  $D^b(Y)$ . Moreover we denote by  $\mathrm{Eq}^{\mathrm{FM}}(D^b(X), D^b(Y))$  the subset of  $\mathrm{Eq}(D^b(X), D^b(Y))$  containing all equivalences of Fourier–Mukai type.

The projective case is very special, as stated in the following deep result due to Orlov (see the main result in [73] and have a look at [22,48] for more general statements).

**Theorem 4.21** ([73]). *Let  $X$  and  $Y$  be smooth projective varieties. Then any equivalence  $\Phi : D^b(X) \xrightarrow{\sim} D^b(Y)$  is of Fourier–Mukai type and the kernel is uniquely determined up to isomorphism.*

Going back to the case of non-algebraic  $K3$  surfaces, the situation immediately becomes more complicated. Indeed, reasoning as in the proof of [43, Cor. 5.23, Cor. 5.24] and using Theorem 4.2, we can prove the following result.

**Proposition 4.22.** *Let  $X$  and  $Y$  be non-isomorphic  $K3$  surfaces such that  $\rho(X) = \rho(Y) = 0$ . Then  $\mathrm{Eq}(D^b(X), D^b(Y)) \neq \emptyset$  and  $\mathrm{Eq}^{\mathrm{FM}}(D^b(X), D^b(Y)) \neq \mathrm{Eq}(D^b(X), D^b(Y))$ .*

**Proof.** By Theorem 4.2, there exists an equivalence  $\Psi : \mathbf{Coh}(X) \xrightarrow{\sim} \mathbf{Coh}(Y)$  which naturally extends by Proposition 4.15 to an equivalence  $\mathcal{T} : D^b(X) \xrightarrow{\sim} D^b(Y)$ . If  $\mathrm{Eq}^{\mathrm{FM}}(D^b(X), D^b(Y)) = \mathrm{Eq}(D^b(X), D^b(Y))$ , then  $\mathcal{T}$  is of Fourier–Mukai type and there exists  $\mathcal{E} \in D^b(X \times Y)$  and an isomorphism of functors

$$\mathcal{T} \cong \Phi_{\mathcal{E}} := \mathbf{R}p_*(\mathcal{E} \otimes^{\mathbf{L}} q^*(-)).$$

Since the minimal objects of  $\mathbf{Coh}(X)$  are the skyscraper sheaves  $\mathcal{O}_x$ , where  $x$  is a closed point of  $X$  (Example 4.12),  $\mathcal{T}$  maps skyscraper sheaves to skyscraper sheaves. Therefore  $\mathcal{E}|_{\{x\} \times Y}$  is isomorphic to a skyscraper sheaf and we naturally get a morphism  $f : X \rightarrow Y$  and  $L \in \text{Pic}(X)$  such that  $\mathcal{T} \cong L \otimes f_*(-)$ . The morphism  $f$  is an isomorphism since  $\Psi$  is an equivalence [43, Cor. 5.22]. In particular  $X$  and  $Y$  would be isomorphic.  $\square$

Actually in the last section we will show that if  $X \not\cong Y$  then  $\text{Eq}^{\text{FM}}(D^b(X), D^b(Y)) = \emptyset$ .

As before, if  $X$  is a K3 surface such that  $\rho(X) = 0$ , we denote by  $\text{Aut}(D^b(X))$  the group of autoequivalences of  $D^b(X)$  and by  $\text{Aut}^{\text{FM}}(D^b(X))$  the subgroup of  $\text{Aut}(D^b(X))$  containing all Fourier–Mukai autoequivalences.

In the next few pages, to shorten the notations we will often write a functor and its derived version in the same way (e.g. we write  $f_*$  instead of  $\mathbf{R}f_*$ ).

#### 4.5. Serre functor

For a compact complex manifold  $X$ , the functor

$$\begin{aligned} S_X : D^b(X) &\longrightarrow D^b(X) \\ \mathcal{E} &\longmapsto \mathcal{F} \otimes \omega_X[\dim(X)], \end{aligned} \tag{4.6}$$

where  $\omega_X$  is the canonical bundle of  $X$ , is called *Serre functor*. The main property of this functor is the existence of the isomorphism of  $\mathbb{C}$ -vector spaces

$$\text{Hom}_{D^b(X)}(\mathcal{E}, \mathcal{F}) \cong \text{Hom}_{D^b(X)}(\mathcal{F}, S_X(\mathcal{E}))^\vee$$

which is functorial in  $\mathcal{E}$  and  $\mathcal{F}$  (see [43, Thm. 3.12] for the projective case and [9, Prop. 5.1.1] for the general case).

Since in  $D^b(X)$  and  $D^b(Y)$ , the Hom’s are finite-dimensional complex vector spaces, any equivalence  $F : D^b(X) \rightarrow D^b(Y)$  (not necessarily of Fourier–Mukai type) commutes with  $S_X$  and  $S_Y$ , i.e. there exists an isomorphism  $S_Y \circ F \cong F \circ S_X$ . Indeed, since  $F$  is fully faithful,

$$\text{Hom}(\mathcal{F}, S_X(\mathcal{G})) = \text{Hom}(F(\mathcal{F}), F(S_X(\mathcal{G}))) \quad \text{and} \quad \text{Hom}(\mathcal{G}, \mathcal{F}) = \text{Hom}(F(\mathcal{G}), F(\mathcal{F})),$$

for any  $\mathcal{F}, \mathcal{G} \in D^b(X)$ . Moreover,

$$\text{Hom}(\mathcal{F}, S_X(\mathcal{G})) = \text{Hom}(\mathcal{G}, \mathcal{F})^\vee \quad \text{and} \quad \text{Hom}(F(\mathcal{G}), F(\mathcal{F})) = \text{Hom}(F(\mathcal{F}), S_Y(F(\mathcal{G})))^\vee.$$

Hence, we get a functorial (in  $\mathcal{F}$  and  $\mathcal{G}$ ) isomorphism

$$\text{Hom}(F(\mathcal{F}), S_Y(F(\mathcal{G}))) \cong \text{Hom}(F(\mathcal{F}), F(S_X(\mathcal{G}))).$$

Using the fact that  $F$  is an equivalence, we get an isomorphism between the representable functors  $\text{Hom}(-, S_Y(F(\mathcal{G})))$  and  $\text{Hom}(-, F(S_X(\mathcal{G})))$  and hence an isomorphism  $S_Y(F(\mathcal{G})) \cong F(S_X(\mathcal{G}))$  (see [18]).

**Lemma 4.23.** *Let  $X$  be a K3 surface and let  $Y$  be a compact complex manifold with an equivalence*

$$F : D^b(X) \xrightarrow{\sim} D^b(Y).$$

*Then  $Y$  is a surface with trivial canonical bundle.*

**Proof.** The order of the dualizing sheaf  $\omega_X$  of a compact complex manifold  $X$  is the minimal  $k \in \mathbb{N} \cup \{+\infty\}$  such that  $\omega_X^{\otimes k} \cong \mathcal{O}_X$ . Following the proof of [43, Prop. 4.1] and using the property of the Serre functor previously described, we can show that  $\dim X = \dim Y$  and that the orders of  $\omega_X$  and  $\omega_Y$  are the same.  $\square$

Hence, by the Kodaira classification (see [4]),  $Y$  is either a K3 surface or a two-dimensional complex torus or a Kodaira surface.

**Proposition 4.24.** *If  $Y$  is also assumed to be Kähler, then  $Y$  is a K3 surface with the same Picard number of  $X$ .*

**Proof.** By Example 4.13 we know that  $\mathcal{O}_X$  is a spherical object of  $D^b(X)$ . It is enough to prove that a two-dimensional complex torus  $T$  does not contain spherical objects.

Following for example [13, Lemma 15.1], suppose that  $\mathbf{Coh}(T)$  contains a spherical sheaf  $\mathcal{E}$  of positive rank. Since  $\mathcal{E}^\vee \otimes \mathcal{E} = \mathcal{O}_T \oplus \mathcal{E}nd_0(\mathcal{E})$  (here  $\mathcal{E}nd_0(\mathcal{E})$  is the sheaf of endomorphisms with trace 0),  $\text{Hom}(\mathcal{E}, \mathcal{E}[1]) = \text{Ext}^1(\mathcal{E}, \mathcal{E}) = H^1(T, \mathcal{O}_T) \oplus H^1(T, \mathcal{E}nd_0(\mathcal{E})) \neq 0$ , because  $T$  is a torus and  $H^1(T, \mathcal{O}_T) \cong \mathbb{C}$ . Moreover, any torsion sheaf  $\mathcal{T}$  can be deformed. Thus  $\text{Ext}^1(T, \mathcal{T}) \neq 0$  and  $T$  is not spherical.

Now let  $\mathcal{E}$  be a spherical object in  $D^b(T)$  and let  $r$  be the maximal integer such that  $H^r(\mathcal{E}) \neq 0$ . There exists a spectral sequence

$$E_2^{p,q} = \bigoplus_{i-j=p} \text{Ext}_{\mathbf{Coh}(Y)}^q(H^i(\mathcal{E}), H^j(\mathcal{E})) \implies \text{Hom}_{D^b(Y)}(\mathcal{E}, \mathcal{E}[p+q]),$$

where  $H^j(\mathcal{E})$  is the  $j$ th cohomology sheaf of  $\mathcal{E}$ . For  $q = 1, p = 0$  and  $i = j = 1$  there is some element surviving in  $\text{Hom}(\mathcal{E}, \mathcal{E}[1])$  and so  $\mathcal{E}$  cannot be spherical.

Hence  $Y$  is a  $K3$  surface. Let  $\mathcal{N}(X) := K(X)/K(X)^\perp$  and  $\mathcal{N}(Y) := K(Y)/K(Y)^\perp$ , where the orthogonal is taken with respect to the Euler pairing (4.5). The existence of the Chern character and the Hirzebruch–Riemann–Roch Theorem give the identifications  $\mathcal{N}(X) = H^0(X, \mathbb{Z}) \oplus \text{NS}(X) \oplus H^4(X, \mathbb{Z})$  and  $\mathcal{N}(Y) = H^0(Y, \mathbb{Z}) \oplus \text{NS}(Y) \oplus H^4(Y, \mathbb{Z})$ . Clearly an exact equivalence  $D^b(X) \cong D^b(Y)$  induces an isometry (with respect to the Euler pairing)  $\mathcal{N}(X) \cong \mathcal{N}(Y)$ .  $\square$

- Remark 4.25.** (i) We will see in the next section (Corollary 5.3) that the previous proposition is general: all Fourier–Mukai partners of a  $K3$  surface  $X$  are  $K3$  surfaces, with the same Picard number of  $X$ . Nevertheless, Proposition 4.24 is interesting in itself. Indeed, it works also for equivalences not of Fourier–Mukai type.
- (ii) As an easy consequence, we deduce that  $X$  is projective if and only if all  $K3$  surfaces  $Y$  with  $D^b(X) \cong D^b(Y)$  are projective. Indeed, the signatures of  $\mathcal{N}(X)$  and  $\mathcal{N}(Y)$  are the same. A more precise picture emerges from the recent paper [85] which we will not discuss here.

### 5. Autoequivalences

We are now ready to move to the second main problem that we want to discuss: the description of the group of autoequivalences of Fourier–Mukai type of the derived category of a generic analytic  $K3$  surface.

The first step will be to study the action induced by any equivalence of Fourier–Mukai type either on the Grothendieck group or on the Mukai lattice. Secondly we will use the machinery of stability conditions introduced by Bridgeland to produce a complete description.

#### 5.1. Action on $K$ -theory and cohomology

In a first step, let  $X$  and  $Y$  be compact complex manifolds. Let  $\Phi_{\mathcal{E}} : D^b(X) \rightarrow D^b(Y)$  be a Fourier–Mukai equivalence with kernel  $\mathcal{E} \in D^b(X \times Y)$ . Then the induced isomorphism at the level of Grothendieck groups is given by the morphism  $\Phi_{[\mathcal{E}]}^K : K(X) \rightarrow K(Y)$  defined by

$$\Phi_{[\mathcal{E}]}^K(e) := q_*([\mathcal{E}] \cdot p^*(e)),$$

where  $p : X \times Y \rightarrow X$  and  $q : X \times Y \rightarrow Y$  are the natural projections. In other words, the following diagram commutes

$$\begin{CD} D^b(X) @>\Phi_{\mathcal{E}}>> D^b(Y) \\ @V[-]VV @VV[-]V \\ K(X) @>\Phi_{[\mathcal{E}]}^K>> K(Y). \end{CD} \tag{5.1}$$

Using the definition of Mukai vector, our analysis of Fourier–Mukai equivalences of the bounded derived categories can be further developed. Indeed, for  $\mathcal{E} \in D^b(X)$ , one can easily consider the Mukai vector  $v([\mathcal{E}])$  of  $\mathcal{E}$  just by means of Definition 4.5. Now the morphism  $\Phi_{[\mathcal{E}]}^K : K(X) \rightarrow K(Y)$  gives rise to a map  $\Phi_{v([\mathcal{E}])}^H : H^*(X, \mathbb{Q}) \rightarrow H^*(Y, \mathbb{Q})$  such that

$$\Phi_{v([\mathcal{E}])}^H : b \longmapsto q_*(v([\mathcal{E}]) \cdot p^*(b)).$$



The Grothendieck–Riemann–Roch Theorem shows that the following diagram commutes:

$$\begin{array}{ccc}
 K(X) & \xrightarrow{\Phi_{[\mathcal{E}]}^K} & K(Y) \\
 v(-) \downarrow & & \downarrow v(-) \\
 H^*(X, \mathbb{Q}) & \xrightarrow{\Phi_{v([\mathcal{E}])}^H} & H^*(Y, \mathbb{Q}).
 \end{array} \tag{5.2}$$

From now on, given a Fourier–Mukai equivalence  $\Phi_{\mathcal{E}} : D^b(X) \rightarrow D^b(Y)$ , we denote by

$$\Phi_{\mathcal{E}}^H : H^*(X, \mathbb{Q}) \longrightarrow H^*(Y, \mathbb{Q})$$

the morphism induced on cohomology via the diagrams (5.1) and (5.2) and, to simplify the notation, we write  $v(\mathcal{E}) := v([\mathcal{E}])$ , for any  $\mathcal{E} \in D^b(X)$ .

**Lemma 5.1.** *Let  $X$  and  $Y$  be compact complex manifolds and let  $\Phi_{\mathcal{F}} : D^b(X) \rightarrow D^b(Y)$  be a Fourier–Mukai equivalence whose kernel is  $\mathcal{F} \in D^b(X \times Y)$ . Then*

$$\Phi_{\mathcal{F}}^H : H^*(X, \mathbb{Q}) \longrightarrow H^*(Y, \mathbb{Q})$$

is an isomorphism of  $\mathbb{Q}$ -vector spaces.

**Proof.** The proof in the non-algebraic case follows line-by-line [43, Prop. 5.33]. Since it is quite easy, we reproduce it here for the convenience of the reader. For simplicity, we put  $\Phi := \Phi_{\mathcal{F}}$  and  $\Phi^H := \Phi_{\mathcal{F}}^H$ .

The  $\mathbb{Q}$ -linearity is clear from the definition of  $\Phi^H$ . So it is enough to prove that  $\Phi^H$  is bijective. Since  $\Phi$  is an equivalence, its right and left adjoints coincide with the inverses. Moreover,  $\Phi \circ \Phi^{-1} = \text{id} = \Phi_{\mathcal{O}_{\Delta}}$ , where  $\Delta$  is the image of the diagonal embedding  $i : \Delta \hookrightarrow X \times X$ . Indeed, for any  $\mathcal{E} \in D^b(X)$ ,

$$\begin{aligned}
 \Phi_{\mathcal{O}_{\Delta}}(\mathcal{E}) &= p_*(\mathcal{O}_{\Delta} \otimes q^*(\mathcal{E})) \\
 &= p_*(i_*\mathcal{O}_X \otimes q^*(\mathcal{E})) \\
 &= p_*(i_*(\mathcal{O}_X \otimes i^*q^*(\mathcal{E}))) \\
 &= (p \circ i)_*(q \circ i)^*\mathcal{E} \\
 &= \mathcal{E},
 \end{aligned}$$

where  $p$  and  $q$  are the natural projections, all the functors are supposed to be derived and the second equality is due to the Projection Formula (see Section 4.4).

Hence it suffices to prove that  $\Phi_{\mathcal{O}_{\Delta}}^H = \text{id}$ . Since  $\text{ch}(\mathcal{O}_X) = (1, 0, 0)$  and the diagonal is isomorphic to  $X$ , using the Grothendieck–Riemann–Roch Theorem we get

$$\text{ch}(\mathcal{O}_{\Delta}) \cdot \text{td}(X \times X) = i_*(\text{td}(X)).$$

Dividing by  $\sqrt{\text{td}(X \times X)}$  and using the Projection Formula, we get

$$\text{ch}(\mathcal{O}_{\Delta}) \cdot \sqrt{\text{td}(X \times X)} = i_* (\text{td}(X) \cdot i^*((\sqrt{\text{td}(X \times X)})^{-1})).$$

Since  $i^*\text{td}(X \times X) = \text{td}(X)^2$ ,  $i_*(\text{td}(X) \cdot i^*((\sqrt{\text{td}(X \times X)})^{-1})) = i_*(1, 0, 0)$  and thus

$$\begin{aligned}
 \Phi_{\mathcal{O}_{\Delta}}^H(b) &= p_*(\text{ch}(\mathcal{O}_{\Delta}) \cdot \sqrt{\text{td}(X \times X)} \cdot q^*(b)) \\
 &= p_*(i_*(1) \cdot q^*(b)) \\
 &= p_*(i_*(i^*(\cdot q^*(b)))) \\
 &= b.
 \end{aligned}$$

The penultimate equality is given by the Projection Formula, while the last one is trivially due to the fact that  $p \circ i = q \circ i = \text{id}$ .  $\square$

Lemma 5.1 can be made more precise for  $K3$  surfaces.

**Theorem 5.2.** *Let  $X$  and  $Y$  be K3 surfaces and let  $\Phi_{\mathcal{F}} : D^b(X) \rightarrow D^b(Y)$  be a Fourier–Mukai equivalence whose kernel is  $\mathcal{F} \in D^b(X \times Y)$ . Then  $\Phi_{\mathcal{F}}^H$  restricts to a Hodge isometry*

$$\Phi_{\mathcal{F}}^H : \tilde{H}(X, \mathbb{Z}) \longrightarrow \tilde{H}(Y, \mathbb{Z}).$$

**Proof.** The complete proof of this result (originally due to Orlov, [73]) is in [43]. Hence we will simply sketch the main steps. As before we put  $\Phi := \Phi_{\mathcal{F}}$  and  $\Phi^H := \Phi_{\mathcal{F}}^H$ .

First notice that, the isomorphism  $\Phi^H$  defined over  $\mathbb{Q}$  (Lemma 5.1) restricts to integral coefficients by the integrality of the Chern character as discussed in Section 4.4. This is an easy exercise which we leave to the reader.

To prove that  $\Phi_{\mathcal{F}}^H$  preserves the Hodge structure defined in Section 2.3, it is enough to show that  $\Phi_{\mathcal{F}}^H(H^{p,q}(X)) \subset \bigoplus_{r-s=p-q} H^{r,s}(X)$ . The Künneth decomposition of  $c := \text{ch}(\mathcal{F}) \cdot \sqrt{\text{td}(X \times Y)}$  is  $\sum \alpha_{u,v} \boxtimes \beta_{r,s}$ , where  $\alpha_{u,v} \in H^{u,v}(X)$  and  $\beta_{r,s} \in H^{r,s}(Y)$ . Since  $c$  is algebraic, the non-trivial terms are the ones such that  $u + r = v + s$ . If  $\alpha \in H^{p,q}(X)$ , then

$$\Phi_{\mathcal{F}}^H(\alpha) = \sum \left( \int_X (\alpha \wedge \alpha_{u,v}) \right) \beta_{r,s} \in \bigoplus H^{r,s}(Y).$$

In particular the non-zero terms satisfy the condition  $u + p = v + q = \dim X$ . Thus  $p - q = v - u = r - s$ .  $\square$

This is enough to generalize Proposition 4.24.

**Corollary 5.3.** *If  $X$  is a K3 surface and  $Y$  is a smooth compact complex variety which is a Fourier–Mukai partner of  $X$ , then  $Y$  is a K3 surface and  $\rho(X) = \rho(Y)$ .*

**Proof.** As a consequence of Lemma 4.23, we have already proved that  $Y$  is either a K3 surface or a two-dimensional complex torus or a Kodaira surface. To exclude the last two possibilities observe that if  $Y$  is a torus or a Kodaira surface, then  $\dim H^*(X, \mathbb{Q}) \neq \dim H^*(Y, \mathbb{Q})$ . But this would contradict Lemma 5.1.

The Picard numbers of  $X$  and  $Y$  are the same because, by Theorem 5.2, any equivalence  $\Phi : D^b(X) \xrightarrow{\sim} D^b(Y)$  is such that  $\Phi^H(H^{2,0}(X)) = \Phi^H(\tilde{H}^{2,0}(X)) = \tilde{H}^{2,0}(Y) = H^{2,0}(Y)$ . This yields an isometry  $T(X) \cong T(Y)$ .  $\square$

In the generic non-algebraic case the following holds:

**Corollary 5.4.** *Let  $X$  and  $Y$  be generic analytic K3 surfaces which are Fourier–Mukai partners. Then  $X \cong Y$ .*

**Proof.** By the previous result an equivalence  $\Phi_{\mathcal{E}} : D^b(X) \xrightarrow{\sim} D^b(Y)$  of Fourier–Mukai type induces a Hodge isometry  $\tilde{H}(X, \mathbb{Z}) \cong \tilde{H}(Y, \mathbb{Z})$ . The facts that the weight-two Hodge structures are preserved and that  $T(X) = H^2(X, \mathbb{Z})$  and  $T(Y) = H^2(Y, \mathbb{Z})$  imply that there is a Hodge isometry  $H^2(X, \mathbb{Z}) \cong H^2(Y, \mathbb{Z})$ . Theorem 2.12 yields now the desired isomorphism  $X \cong Y$ .  $\square$

**Remark 5.5.** As a consequence, a generic analytic K3 surface does not have non-isomorphic Fourier–Mukai partners. On the other hand, the algebraic case is very much different. In fact, in [69] and [83] it is proved that for any positive integer  $N$ , there are  $N$  non-isomorphic algebraic K3 surfaces  $X_1, \dots, X_N$  such that  $D^b(X_i) \cong D^b(X_j)$ , for  $i, j \in \{1, \dots, N\}$ .

Let us develop further the consequences of Theorem 5.2. Indeed, consider a (not necessarily projective) K3 surface  $X$ . If  $O_{\text{Hodge}}(\tilde{H}(X, \mathbb{Z}))$  denotes the set of Hodge isometries of the Mukai lattice  $\tilde{H}(X, \mathbb{Z})$ , then Theorem 5.2 yields a group homomorphism

$$\Pi : \text{Aut}^{\text{FM}}(D^b(X)) \longrightarrow O_{\text{Hodge}}(\tilde{H}(X, \mathbb{Z})).$$

Of course, information about the image of  $\Pi$  would give interesting properties of  $\text{Aut}^{\text{FM}}(D^b(X))$ . Notice that a complete description of the group of autoequivalences in terms of a group homomorphism analogous to  $\Pi$  is available for the closest relatives of K3 surfaces: abelian varieties [74].

Nevertheless, the situation for K3 surfaces is widely unknown and the expectation is summarized by the following conjecture.

**Conjecture 5.6.** For a K3 surface  $X$ , the image of  $\Pi$  coincides with the subgroup

$$O_+(\tilde{H}(X, \mathbb{Z})) \subseteq O_{\text{Hodge}}(\tilde{H}(X, \mathbb{Z}))$$

of orientation preserving Hodge isometries.

This conjecture was first stated by Szendrői in [84], who has seen it as a “mirror” version of a result of Borcea [11] and Donaldson [26] about the orientation preserving property for diffeomorphisms of K3 surfaces. Bridgeland [13] improved it, by proposing a description of the kernel of  $\Pi$  in terms of the fundamental group of a domain naturally associated to a manifold parametrizing stability conditions on the triangulated category  $D^b(X)$  (see Section 5.2). In the algebraic case, some evidence for the truth of Conjecture 5.6 has been found in [44] (see also [40]). The generic twisted case is proved in [45]. In Section 5.3 the conjecture will be proved for generic analytic K3 surface following the proof in [45]. More recently, in [46], the conjecture was proved for smooth projective (untwisted) K3 surfaces.

- Example 5.7.** (i) The Hodge isometry induced on cohomology by the shift functor  $[1] : D^b(X) \xrightarrow{\sim} D^b(X)$  is  $-\text{id} : \tilde{H}(X, \mathbb{Z}) \xrightarrow{\sim} \tilde{H}(X, \mathbb{Z})$  which is orientation preserving.  
 (ii) Let  $f : X \xrightarrow{\sim} Y$  be an isomorphism of K3 surfaces and consider the Fourier–Mukai equivalence  $f^* : D^b(Y) \xrightarrow{\sim} D^b(X)$ . Obviously,  $(f^*)^H = f^*|_{H^2(X, \mathbb{Z})} \oplus \text{id}_{H^0(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})}$  which preserves the natural orientation of the four positive directions (Example 2.14).  
 (iii) If  $X$  is a K3 surface, the sheaf  $\mathcal{O}_X$  is a spherical object (see Example 4.13) and the isometry induced on cohomology by the spherical twist  $T_{\mathcal{O}_X}$  (Example 4.19) is the reflection with respect to the vector  $(1, 0, 1) \in H^*(X, \mathbb{Z})$  described in Example 2.14 which preserves the orientation.

Using the previous examples we can prove the following:

**Proposition 5.8.** Let  $X$  be a generic analytic K3 surface and let  $g \in O_+(\tilde{H}(X, \mathbb{Z}))$ . Then there exists  $\Phi \in \text{Aut}^{\text{FM}}(D^b(X))$  such that  $g = \Pi(\Phi)$ .

**Proof.** From the very definition of orientation preserving Hodge isometries and using the fact that  $\rho(X) = \rho(Y) = 0$ , it follows easily that  $g = h_1 \circ h_2$ , where  $h_1 = k_1 \oplus \text{id}_{H^2(X, \mathbb{Z})}$  and  $h_2 = \text{id}_{H^0(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})} \oplus k_2$ , for  $k_1 \in O(H^0(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}))$  and  $k_2 \in O(H^2(X, \mathbb{Z}))$ . Up to composing  $g$  with  $-\text{id}_{\tilde{H}(X, \mathbb{Z})}$  (which, by Example 5.7 (i), is induced by the shift functor) we can also suppose that  $h_1$  and  $h_2$  are orientation preserving.

In this situation the Torelli Theorem immediately yields an automorphism  $f : X \xrightarrow{\sim} X$  such that  $h_2 = f^*$ . On the other hand,  $h_1$  is either the identity or the reflection with respect to the vector  $(1, 0, 1)$ . The second possibility is realized by the spherical twist (Example 5.7(iii)).  $\square$

### 5.2. Stability conditions

Stability conditions were introduced by Bridgeland in [12] to deal with Douglas’ notion of  $\Pi$ -stability [2,27]. Nevertheless, the relevance of stability conditions and the topology of the manifolds parametrizing them became immediately apparent in the geometric context as well. In particular, stability conditions defined on the bounded derived categories of coherent sheaves on complex varieties were studied in many interesting geometric cases (see, for example, [13,14,56,71,72]). Very nice surveys for this are [15,16].

As we mentioned in the previous section, Bridgeland suggested in [13] that a possible strategy to prove Conjecture 5.6 would involve the topology of the manifold parametrizing stability conditions on the derived categories of K3 surfaces. In this paper, we will not deal with these topological problems since in Section 5.3 we will see that the study of the equivalences of the derived categories of generic analytic K3 surface just relies on the notion of stability condition and does not involve any topological property.

Since, due to Proposition 4.15, there exists a natural equivalence  $D^b(X) \cong D^b(\mathbf{Coh}(X))$ , throughout this section we will not distinguish between these two categories. Moreover  $X$  will be a generic analytic K3 surface and  $\mathcal{N}(X)$  will denote the lattice  $H^0(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}) \cong U$ . We will restrict our attention to stability conditions of special type.

**Definition 5.9.** A (numerical) stability condition on  $D^b(X)$  is a pair  $\sigma = (Z, \mathcal{P})$  where

$$Z : \mathcal{N}(X) \longrightarrow \mathbb{C}$$

is a linear map (the *central charge*) while  $\mathcal{P}(\phi) \subset D^b(X)$  are full additive subcategories for each  $\phi \in \mathbb{R}$  satisfying the following conditions:

- (a) If  $0 \neq \mathcal{E} \in \mathcal{P}(\phi)$ , then  $Z(\mathcal{E}) = m(\mathcal{E}) \exp(i\pi\phi)$  for some  $m(\mathcal{E}) \in \mathbb{R}_{>0}$ .
- (b)  $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$  for all  $\phi$ .
- (c)  $\text{Hom}_{D^b(X)}(\mathcal{E}_1, \mathcal{E}_2) = 0$  for all  $\mathcal{E}_i \in \mathcal{P}(\phi_i)$  with  $\phi_1 > \phi_2$ .
- (d) Any  $0 \neq \mathcal{E} \in D^b(X)$  admits a *Harder–Narasimhan filtration* given by a collection of distinguished triangles  $\mathcal{E}_{i-1} \rightarrow \mathcal{E}_i \rightarrow \mathcal{A}_i$  with  $\mathcal{E}_0 = 0$  and  $\mathcal{E}_n = \mathcal{E}$  such that  $\mathcal{A}_i \in \mathcal{P}(\phi_i)$  with  $\phi_1 > \dots > \phi_n$ .

Following Bridgeland’s terminology, the objects in the category  $\mathcal{P}(\phi)$  are called *semistable* of phase  $\phi$  while the objects  $\mathcal{A}_i$  in (d) are called the *semistable factors* of  $\mathcal{E}$ . It is a simple exercise to show that they are unique up to isomorphism. The minimal objects of  $\mathcal{P}(\phi)$  are called *stable* of phase  $\phi$ .

Given a stability condition  $(Z, \mathcal{P})$  and  $\mathcal{E} \in D^b(X)$ , we put  $\phi_-(\mathcal{E}) := \phi_1$  and  $\phi_+(\mathcal{E}) := \phi_n$ , where  $\phi_1$  and  $\phi_n$  are the phases of the first and the last semistable factor in the Harder–Narasimhan filtration of  $\mathcal{E}$  (see item (d) of the previous definition). One defines  $\mathcal{P}((0, 1])$  to be the subcategory of  $D^b(X)$  containing the zero object and all  $\mathcal{E} \in D^b(X)$  such that  $0 < \phi_-(\mathcal{E}) \leq \phi_+(\mathcal{E}) \leq 1$ .

- Remark 5.10.** (i) All stability conditions studied in this section are tacitly assumed to be *locally finite* (see [13] for the precise definition). For our purposes it is enough to know that this additional condition implies that any semistable object  $\mathcal{E} \in \mathcal{P}(\phi)$  admits a finite Jordan–Hölder filtration, i.e. a finite filtration  $0 = \mathcal{E}_0 \subset \dots \subset \mathcal{E}_n = \mathcal{E}$  with stable quotients  $\mathcal{E}_{i+1}/\mathcal{E}_i \in \mathcal{P}(\phi)$ .
- (ii) There is an alternative way to give a stability condition on  $D^b(X)$ . Indeed [12, Prop. 5.3] shows that a stability condition can equivalently be described by a bounded *t*-structure on  $D^b(X)$  with heart  $\mathbf{A}$  and a stability function  $Z : K(\mathbf{A}) \rightarrow \mathbb{C}$  with Harder–Narasimhan filtrations (recall that a stability function  $Z$  is a group homomorphism such that  $Z(E) \in \mathbb{H} \cup \mathbb{R}_{<0}$ , for all  $0 \neq E \in \mathbf{A}$ , where  $\mathbb{H}$  is the complex upper half-plane). In particular, given a stability condition  $\sigma = (Z, \mathcal{P})$ , one can prove that  $\mathbf{A} := \mathcal{P}((0, 1])$  is the heart (usually called the *heart of  $\sigma$* ) of a bounded *t*-structure.

**Definition 5.11.** The space of all locally finite numerical stability conditions on  $D^b(X)$  is denoted by  $\text{Stab}(D^b(X))$ .

In general, it is a highly non-trivial task to show that the space parametrizing stability conditions on a triangulated category is non-empty but, once this is achieved, [12, Thm. 1.2] shows that such a space is a manifold. Moreover, if the triangulated category is the bounded derived category of coherent sheaves on some complex manifold, then the space parametrizing stability conditions is actually a finite-dimensional complex manifold [12, Cor. 1.3].

Hence, let us first show that, in our specific case,  $\text{Stab}(D^b(X))$  is non-empty. Consider the open subset

$$R := \mathbb{C} \setminus \mathbb{R}_{\geq -1} = R_+ \cup R_- \cup R_0, \tag{5.3}$$

where  $R_+ := R \cap \mathbb{H}$ ,  $R_- := R \cap (-\mathbb{H})$  and  $R_0 := R \cap \mathbb{R}$  with  $\mathbb{H}$  denoting the upper half-plane. Given  $z = u + iv \in R$ , following [13], the subcategories  $\mathcal{F}(z), \mathcal{T}(z) \subset \mathbf{Coh}(X)$  are defined as follows:

- $\mathcal{F}(z)$  is the full subcategory of all torsion free sheaves of degree  $\leq v$ ;
- $\mathcal{T}(z)$  is the full subcategory that contains all torsion sheaves and all torsion free sheaves of degree  $> v$ .

Therefore, if  $z \in R_+ \cup R_0$ ,  $\mathcal{F}(z)$  and  $\mathcal{T}(z)$  are respectively the full subcategories of all torsion free sheaves respectively torsion sheaves while, if  $z \in R_-$ ,  $\mathcal{F}(z)$  is trivial and  $\mathcal{T}(z) = \mathbf{Coh}(X)$ .

Consider the subcategories defined by means of  $\mathcal{F}(z)$  and  $\mathcal{T}(z)$  as follows. If  $z \in R_+ \cup R_0$ , we put

$$\mathcal{A}(z) := \left\{ \mathcal{E} \in D^b(X) : \begin{array}{l} \bullet H^i(\mathcal{E}) = 0 \text{ if } i \notin \{-1, 0\} \\ \bullet H^{-1}(\mathcal{E}) \in \mathcal{F}(z) \\ \bullet H^0(\mathcal{E}) \in \mathcal{T}(z) \end{array} \right\}.$$

On the other hand, let  $\mathcal{A}(z) = \mathbf{Coh}(X)$  when  $z \in R_-$ . Due to [13, Lemma 3.3],  $\mathcal{A}(z)$  is the heart of a bounded *t*-structure for any  $z \in R$ .

Now, for any  $z = u + iv \in R$  we define the function

$$\begin{aligned} Z : \mathcal{A}(z) &\longrightarrow \mathbb{C} \\ \mathcal{E} &\longmapsto \langle v(\mathcal{E}), (1, 0, z) \rangle = -u \cdot r - s - i(r \cdot v), \end{aligned}$$

where  $v(\mathcal{E}) = (r, 0, s) = (r, 0, r - c_2(\mathcal{E}))$  is the Mukai vector of  $\mathcal{E}$ .

The main properties of the pair  $(\mathcal{A}(z), Z)$  just defined are summarized by the following proposition.

**Proposition 5.12** ([45], Lemma 3.2 and Proposition 3.4). *For any  $z \in R$  the function  $Z$  defines a stability function on  $\mathcal{A}(z)$  which has the Harder–Narasimhan property.*

Using item (ii) of Remark 5.10, we conclude that for any  $z \in R$  we have a stability condition  $\sigma_z$  obtained from the  $t$ -structure associated with the pair  $(\mathcal{F}(z), \mathcal{T}(z))$  and the stability function  $Z$  on  $\mathcal{A}(z)$ . In [45, Prop. 3.4] it is also proved that such a stability condition is locally finite. Hence  $\sigma_z \in \text{Stab}(D^b(X))$  and  $\text{Stab}(D^b(X))$  is non-empty. Some additional interesting topological properties of  $\text{Stab}(D^b(X))$  (not used in the rest of this paper) are determined by the following result.

**Theorem 5.13** ([45], Theorem 3). *The space  $\text{Stab}(X)$  is connected and simply-connected.*

**Remark 5.14.** As pointed out in [12], there is a left action of the group  $\text{Aut}^{\text{FM}}(X)$  on the manifold  $\text{Stab}(D^b(X))$ . Indeed, given  $\sigma = (Z, \mathcal{P}) \in \text{Stab}(D^b(X))$  and  $\Phi \in \text{Aut}^{\text{FM}}(X)$  we define  $\Phi(\sigma)$  to be the stability condition  $(Z', \mathcal{P}') \in \text{Stab}(D^b(X))$  where  $Z' = Z \circ (\Phi^H)^{-1}$  and  $\mathcal{P}'(\phi) = \Phi(\mathcal{P}(\phi))$ .

Using Lemma 4.16 one can show the following remarkable property of  $\text{Stab}(D^b(X))$  which will be used in Section 5.3.

**Proposition 5.15** ([45], Proposition 1.18 and Corollary 1.19). *For any  $\sigma \in \text{Stab}(D^b(X))$ , there is  $n \in \mathbb{Z}$  such that  $T_{\mathcal{O}_x}^n(\mathcal{O}_x)$  is stable in  $\sigma$ , for any closed point  $x \in X$ .*

### 5.3. Equivalences

For generic analytic K3 surfaces  $X$ , we obtain a complete description of the group of autoequivalences of  $D^b(X)$  using the notion of stability condition. The proof immediately shows that the topology of  $\text{Stab}(X)$  is irrelevant in this case.

**Proposition 5.16** ([45], Lemma 3.9). *Let  $X$  and  $Y$  be complex K3 surfaces such that  $\rho(X) = \rho(Y) = 0$ . If  $\Phi_{\mathcal{E}} : D^b(X) \xrightarrow{\sim} D^b(Y)$  is an equivalence of Fourier–Mukai type, then up to shift*

$$\Phi_{\mathcal{E}} \cong T_{\mathcal{O}_Y}^n \circ f_*$$

for some  $n \in \mathbb{Z}$ . Here,  $T_{\mathcal{O}_Y}$  is the spherical twist with respect to  $\mathcal{O}_Y$  and  $f : X \xrightarrow{\sim} Y$  is an isomorphism.

**Proof.** Suppose  $\sigma = \sigma_{(u,v=0)}$  is one of the distinguished stability conditions constructed in Section 5.2 with  $(u, v) \in R$  and  $v = 0$ . Let  $\tilde{\sigma} := \Phi_{\mathcal{E}}(\sigma)$ .

Denote by  $T$  the spherical twist  $T_{\mathcal{O}_Y}$ . By Proposition 5.15, there exists an integer  $n$  such that all skyscraper sheaves  $\mathcal{O}_x$  are stable of the same phase in the stability condition  $T^n(\tilde{\sigma})$ .

The composition  $\Psi := T^n \circ \Phi_{\mathcal{E}}$ , is again an equivalence of Fourier–Mukai type, i.e.  $\Psi := \Phi_{\mathcal{F}}$ , for some kernel  $\mathcal{F} \in D^b(X \times Y)$ . Moreover it sends the stability condition  $\sigma$  to a stability condition  $\sigma'$  for which all skyscraper sheaves are stable of the same phase.

Up to shifting the kernel  $\mathcal{F}$  sufficiently, we can assume that  $\phi_{\sigma'}(\mathcal{O}_y) \in (0, 1]$  for all closed points  $y \in Y$ . Thus, the heart  $\mathcal{P}((0, 1])$  of the  $t$ -structure associated with  $\sigma'$  (see Remark 5.10), which under  $\Phi_{\mathcal{F}}$  is identified with  $\mathcal{A}(u)$ , contains as stable objects the images  $\Psi(\mathcal{O}_x)$  of all closed points  $x \in X$  and all point sheaves  $\mathcal{O}_y$ .

In [45, Rmk. 3.3] it was observed that the only semi-rigid stable objects in  $\mathcal{A}(u)$  are the skyscraper sheaves. Hence, for all  $x \in X$  there exists a point  $y \in Y$  such that  $\Psi(\mathcal{O}_x) \cong \mathcal{O}_y$ .

This suffices to conclude that the Fourier–Mukai equivalence  $\Psi_{\mathcal{F}}$  is a composition of  $f_*$  for some isomorphism  $f : X \xrightarrow{\sim} Y$  and a line bundle twist (see [43, Cor. 5.23]). But there are no non-trivial line bundles on  $Y$ .  $\square$

This immediately yields the following complete description of all Fourier–Mukai autoequivalences in the generic case.

**Theorem 5.17** ([45], Theorem 4). *If  $X$  is a K3 surface with  $\text{Pic}(X) = 0$ , then*

$$\text{Aut}^{\text{FM}}(D^b(X)) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \text{Aut}(X).$$

*The first two factors are generated respectively by the shift functor and the spherical twist  $T_{\mathcal{O}_X}$ .*

This nice description shows that the interesting (and possibly non-trivial) part of  $\text{Aut}^{\text{FM}}(D^b(X))$  is given by the group of automorphisms of  $X$ . Due to Theorem 3.6, such a group is usually trivial while it is isomorphic to  $\mathbb{Z}$  for countably many generic K3 surfaces characterized by the fact that  $\text{Aut}(X)$  is generated by an automorphism  $f$  such that the induced isometry  $f^*$  on the second cohomology group has minimal polynomial of Salem type. Hence, in the complex analytic setting,  $\text{Aut}^{\text{FM}}(D^b(X))$  is either isomorphic to  $\mathbb{Z}^{\oplus 2}$  (for K3 surfaces not of McMullen type) or isomorphic to  $\mathbb{Z}^{\oplus 3}$  (for the countably many K3 surfaces of McMullen type).

We conclude observing that, together with Proposition 5.8, Theorem 5.17 proves Conjecture 5.6 for analytic generic K3 surfaces. More precisely, there exists a short exact sequence of groups

$$0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \text{Aut}^{\text{FM}}(D^b(X)) \longrightarrow \mathcal{O}_+(\tilde{H}(X, \mathbb{Z})) \longrightarrow 0,$$

where the two copies of  $\mathbb{Z}$  are generated by the shift by 2 and by  $T_{\mathcal{O}_Y}^2$  respectively.

**Remark 5.18.** The group  $\text{Aut}^{\text{FM}}(D^b(X))$  encodes some interesting geometric features of the K3 surface  $X$ . Indeed, if  $\text{Pic}(X) \neq 0$ , this group is larger than the corresponding one for  $X$  generic analytic. Indeed, as we have just seen, the equivalences coming from tensoring with non-trivial line bundles are not present in the latter case.

We have already observed that generic analytic K3 surfaces of McMullen type can be distinguished from the ones not of McMullen type using the previous description of the group of autoequivalences.

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